TOPICS IN NON COMMUTATIVE ALGEBRAIC GEOMETRY AND CONGRUENT ZETA FUNCTIONS (PART V). WEYL ALGEBRAS REVISITED

YOSHIFUMI TSUCHIMOTO

1. A review

Let us first recall the definition. (See Part I for details.)

DEFINITION 1.1. Let k be a field. Let n be a positive integer. A Weyl algebra $A_n(k)$ over a commutative ring k is an algebra over k generated by $2n$ elements $\{\gamma_1, \gamma_2, \dots, \gamma_{2n}\}$ with the "canonical commutation relations"

$$\text{(CCR)} \qquad [\gamma_i, \gamma_j](= \gamma_i\gamma_j - \gamma_j\gamma_i) = h_{ij} \qquad (1 \le i, j \le 2n).$$

Where h is a non-degenerate anti-Hermitian $2n \times 2n$ matrix of the following form.

$$(h_{ij}) = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

Throughout this section, the letter h will always represent the matrix above and the letter $\bar{h}$ will always represent the inverse matrix of h.

1.1. perfect field.

DEFINITION 1.2. A field k of characteristic $p \neq 0$ is called **perfect** if the Frobenius homomorphism

$$k \ni x \to x^p \in k$$

is surjective.

1.2. A matrix representation of Weyl algebra.

From now on in this section we fix a prime number p and we assume k is a field of characteristic p. We will simply write A_n instead of $A_n(k)$.

Let us define operators(matrices) $\{\mu_i\}_{i=1}^{2n}$ acting on p^n-dimensional vector space

$$V = k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots x_n^p)$$

by

$$\left.\begin{aligned} \mu_i &= \text{multiplication by } x_i \\ \mu_{i+n} &= \partial/\partial x_i. \end{aligned}\right\} \qquad i = 1, 2, \dots, n.$$

Let

$$S_n = k[T_1, T_2, \dots, T_{2n-1}, T_{2n}]$$

be a polynomial ring of $2n$-variables over k.

Then we have a faithful representation

$$\Phi : A_n \to M_{p^n}(S_n)$$

of the Weyl algebra A_n by putting

$$\Phi(\gamma_i) = T_i \cdot 1_{p^n} + \mu_i.$$

Furthermore, for any $c = (c_1, c_2, \dots, c_{2n}) \in k^{2n}$, we have by specialization the following representation of A_n.

$$\Phi_c : A_n \to M_{p^n}(k)$$

by putting

$$\Phi_c(\gamma_i) = c_i \cdot 1_{p^n} + \mu_i.$$

We recall also that

LEMMA 1.3. *(Part I, Lemma 7.3) The center Z_n of A_n is isomorphic to a polynomial algebra with $2n$ indeterminates. Namely,*

$$Z_n = k[\gamma_1^p, \gamma_2^p, \dots, \gamma_{2n}^p].$$

LEMMA 1.4. *(Part I, Corollary 7.8) Let k be an algebraically closed field of characteristic $p \neq 0$. Then every finite dimensional irreducible representation $\alpha : A_n(k) \to \operatorname{End}_k(V)$ of $A_n(k)$ is equivalent to a representation Φ_c for some $c \in k^{2n}$.*

LEMMA 1.5. *Let k be a field of characteristic p. If k contains infinite number of elements, then:*

$$\bigcap_{c \in k^{2n}} \operatorname{Ker}(\Phi_c) = 0.$$

Here is another thing we need to know.

LEMMA 1.6. *Φ gives a k-algebra isomorphism*

$$\hat{\Phi} : S_n \otimes_{Z_n} A_n \cong M_{p^n}(S_n).$$

PROOF. One may easily verify that we have a well-defined homomorphism given by

$$\hat{\Phi} : S_n \otimes_{Z_n} A_n \ni f \otimes a \mapsto f\Phi(a) \ni M_{p^n}(S_n).$$

For any $j \in \{1, 2, 3, \dots, 2n\}$, we have

$$\hat{\Phi}(1 \otimes \gamma_j - T_j \otimes 1) = \mu_j.$$

So the image Image($\hat{\Phi}$) contains μ_j. Since we know that $\{\mu_j\}_{j=1}^{2n}$ generates $M_{p^n}(k)$ as k-algebra (Part I, Corollary 7.10) , we conclude that the map $\hat{\Phi}$ is surjective.

Now, both $S_n \otimes_{Z_n} A_n$ and $M_{p^n}(S_n)$ is free S_n-modules of rank p^{2n}. So the map $\hat{\Phi}$ is generically injective. (That means, if we take the quotient field $Q(S_n)$ of S_n and consider

$$1_{Q(S_n)} \otimes \hat{\Phi} : Q(S_n) \otimes_{S_n} (S_n \otimes_{Z_n} A_n) \to M_{p^n}(Q(S_n)).$$

then by an elementary theorem in linear algebra, we see that it is an isomorphism.)

Since S_n (a polynomial algebra over k) is an integral domain, $S_n \otimes_{Z_n} A_n$ is Z_n-torsion free. (That means,

$$S_n \otimes_{Z_n} A_n \to Q(S_n) \otimes_{S_n} (S_n \otimes_{Z_n} A_n)$$

is injective.)

So we see that $\hat{\Phi}$ is injective.

□

1.3. **Algebra endomorphisms and centers of Weyl algebras.**

LEMMA 1.7. *Let k be a field of characteristic $p \neq 0$. For any k-algebra endomorphism ϕ of $A_n(k)$,*

(1) *$\Phi_c \circ \phi$ is a surjective homomorphism for any $c \in k^{2n}$.*
(2) *$\phi(Z_n(k)) \subset Z_n(k)$.*

PROOF. We may assume that k is an algebraically closed field.
(1) The composition $\phi_c = \Phi_c \circ \phi$ is a representation of A_n. By Lemma 1.4 we see that any irreducible sub representation of ϕ_c is equivalent to $\Phi_{\bar{c}}$ for some $\bar{c} \in k^{2n}$. By a dimensional argument, we conclude that ϕ_c itself is equivalent to $\Phi_{\bar{c}}$. (In other words, there exists $g_c \in \mathrm{GL}_n(k)$ such that

$$\phi_c(x) = g_c \Phi_{\bar{c}}(x) g_c^{-1} \qquad (\forall x \in A_n)$$

holds.) Thus ϕ_c is surjective as required.
(2) Let $z \in Z_n$, $x \in A_n$. For any $c \in k^{2n}$, we have (using the same notation as above)

$$[\phi_c(z), \phi_c(x)] = \phi_c([z, x]) = 0$$

Since we know by (1) that ϕ_c is surjective, we see that $\phi_c(z)$ belongs to the center of $M_{p^n}(k)$. In particular for any $y \in A_n$, we have

$$\Phi_c([\phi(z), y]) = [\Phi_c(\phi(z)), \Phi_c(y)] = [\phi_c(z), \Phi_c(y)] = 0.$$

Thus

$$[\phi(z), y] \in \bigcap_c \mathrm{Ker}(\Phi_c) = 0.$$

So $\phi(z) \in Z(A_n) = Z_n$ as required.

Corollary 1.8. *Let $\phi : A_n \to A_n$ be a k-algebra endomorphism of A_n. Then by restriction we obtain a homomorphism*

$$\phi_{Z_n} : Z_n \to Z_n.$$

Furthermore, if the base field k is perfect, then ϕ may be uniquely extended to its p-th root.

$$\phi_{S_n} : S_n \to S_n.$$

In precise, Let us write down ϕ_{Z_n} like

$$\phi_{Z_n}(\gamma_j^p) = \sum_I f_{j,I}(\gamma^p)^I \quad (j = 1, 2, \ldots, 2n).$$

Then ϕ_{S_n} is given by the following formula.

$$\phi_{S_n}(T_j) = \sum_I (f_{j,I})^{1/p} T^I \quad (j = 1, 2, \ldots, 2n).$$

Here comes a geometric interpretation of endomorphisms of Weyl algebras.

Corollary 1.9. *Let k be a perfect field of characteristic $p \neq 0$. Let $\phi : A_n \to A_n$ be a k-algebra endomorphism of A_n. Then we have a matrix valued function $G \in \mathrm{GL}_{p^n}(S_n)$ and a morphism $f : \mathrm{Spec}(S_n) \to \mathrm{Spec}(S_n)$ which enables the following diagram commute.*

(*)
$$\begin{array}{ccc} A_n & \xrightarrow{\phi} & A_n \\ \Phi\downarrow & & \Phi\downarrow \\ M_{p^n}(S_n) & \xrightarrow{\bar\phi} & M_{p^n}(S_n) \end{array}$$

Where $\bar\phi$ is defined as

(**)
$$\bar\phi(x) = G f^*(x) G^{-1}.$$

We may write down the commutative diagram above as the following equation.

$$\Phi(\phi(a)) = G f^*(\Phi(a)) G^{-1}$$

Proof. Let us define

$$\phi_{Z_n} : Z_n \to Z_n$$

and

$$\phi_{S_n} : S_n \to S_n$$

as in the previous Corollary. We put $f = \mathrm{Spec}(\phi_{S_n})$.

We have an well-defined Z_n-algebra homomorphism

$$\phi_{S_n} \otimes \phi : S_n \otimes_{Z_n} A_n \to S_n \otimes_{Z_n} A_n.$$

By using the isomorphism in , we obtain a Z_n-homomorphism

$$\bar{\phi} = \hat{\Phi}(\phi_{S_n} \otimes \phi)\hat{\Phi}^{-1} : M_{p^n}(S_n) \to M_{p^n}(S_n)$$

which is compatible with ϕ in the sense that it satisfies the commutative diagram (*) of the statement.

It remains to prove that the map $\bar{\phi}$ is represented as (**). By pull-back, we obtain an S_n-algebra homomorphism

$$\rho : M_{p^n}(S_n) \cong S_n \otimes_{\phi_{S_n}, S_n} M_{p^n}(S_n) \ni y \otimes x \mapsto y\bar{\phi}(x) \in M_{p^n}(S_n).$$

Where the first isomorphism in the above line is the inverse of the following S_n-algebra homomorphism.

$$S_n \otimes_{\phi_{S_n}, S_n} M_{p^n}(S_n) \ni y \otimes x \mapsto y f^*(x) \in M_{p^n}(S_n).$$

by an argument similar to that in (I,Lemma 7.9), we see that there exists $G \in GL_{p^n}(S_n)$ such that

$$\rho(x) = GxG^{-1} \qquad (\forall y \in M_{p^n}(S_n))$$

holds. (See appendix for the detail.)

□

1.4. **appendix.**

Definition 1.10. (temporary) Let R be a commutative ring. A finitely generated R-module M is said to be **projective** if there exists an R-module M' such that

$$M \oplus M' \cong R^{\oplus n}$$

for some n.

We shall argue homological algebra in much more detail later. So the definition here is meant to be minimum. See a book on homological algebra for "correct definition".

Definition 1.11. Let R be a domain. The generic rank of M is the rank of $Q(R) \otimes_R M$ over the quotient field $Q(R)$.

Lemma 1.12. *Let R be a commutative unique factorization domain(UFD). Suppose an R module M satisfies the following conditions.*

1. *M is of generic rank* 1.
2. *M is finitely generated.*
3. *M is projective.*

Then M is R-free of rank 1.

PROOF. (Essentially borrowed from [1]) Let $K = Q(R)$ be the quotient field of R.

Since M is projective and R is an integral domain, M is torsion free. So

$$\iota : M \to M \otimes_R K$$

is injective. Since M is of generic rank 1, $M \otimes_R K$ is isomorphic to K. as a K-module. We may thus assume that $M \subset K$. Since M is finitely generated, we may further assume that $M \subset R$.

Now, Let us paraphrase the condition that M being projective. First of all, the condition is equivalent to an existence of R-module homomorphisms

$$f : M \to R^n, g : R^n \to M$$

such that $g \circ f = \mathrm{id}$. Secondly, we may then represent f, g in matrix form.

$$f(m) = \begin{pmatrix} f_1(m) \\ f_2(m) \\ \vdots \\ f_n(m) \end{pmatrix}, \qquad g(\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}) = (g_1, g_2, g_3, \dots, g_n) \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

$$f : M \to R^n, g : R^n \to M$$

such that $g \circ f = \mathrm{id}$.

Thirdly, each f_i, g_i is represented by a linear map from K to K. That means, by an element of K.

$$f = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \qquad g = (b_1, b_2, b_3, \dots, b_n)$$

We may obtain several properties of $\{a_i\}, \{b_j\}$:

(i) $g(R^n) \subset M$ implies $b_i \in M \subset R (\forall i)$.
(ii) $g \circ f : R^n \to R^n$ implies $a_i b_j \in R (\forall i \forall j)$.
(iii) $g(R^n) \subset M$ implies $b_i \in M (\forall i)$.
(iv) $g(R^n) = M$ implies $\sum_i R b_i = M$.
(v) $g \circ f = \mathrm{id}$ implies $\sum_i b_i a_i = 1$.

Let us write

$$a_i = \frac{l_i}{m_i} \qquad (\gcd(l_i, m_i) = 1).$$

$$R \ni a_i b_j = \frac{l_i b_j}{m_i}$$

Since l_i and m_i are coprime, we conclude that b_j is divisible by m_i. Let us denote by l the largest common multiple of $m_1, m_2, \dots, m_n$.

$$l | b_j$$

By (iv) we see $M \subset lR$.

By (v) we see

$$l = l \sum_i b_i a_i = \sum_i b_i l_i \frac{l}{m_i} \in \sum_i b_i R \subset M.$$

Thus $M \supset lR$. So $M = lR$. □

LEMMA 1.13. *Let R be a commutative UFD. Then for any R-algebra homomorphism*

$$\rho : M_n(R) \to M_n(R),$$

there exits an element $G \in \mathrm{GL}_n(R)$ such that

$$\rho(x) = GxG^{-1}$$

holds for any $x \in M_n(R)$.

PROOF. Let us denote by $e_{ij} \in M_n(R)$ the matrix element. Let us consider R-modules

$$M_i = \rho(e_{ii}) R^n \qquad (i = 1, 2, 3, \dots, n).$$

Then by an argument similar to that in (I,Lemma 7.9), we see that

$$R^n = \bigoplus_{i=1}^{n} M_i.$$

and that the multiplication by $\rho(e_{ji})$

$$M_i \overset{\rho(e_{ji}).}{\to} M_j$$

give isomorphisms between the modules. Hence we see easily that M_0 satisfy the assumptions of the previous lemma. We conclude that M_1 is freely generated by single element v_1. Then we put

$$v_j = \rho(e_{j1}) v_1.$$

and

$$G = (v_1 v_2 v_3 \dots v_n).$$

We may easily see that G plays the roles as expected. □

2. APPENDIX 2

PROPOSITION 2.1. *Let d be a positive integer. Let (X, ω_X), (Y, ω_Y) be smooth symplectic algebraic varieties of dimension $2n$ over a field k. Let $\phi : X \to Y$ be a symplectic morphism. That means, it is a morphism which preserves the symplectic structure :*

$$\phi^*(\omega_Y) = \omega_X$$

Then the tangent map of ϕ is of full rank at every point P on X.

We make an effective use of the theory of the Pfaffian $\mathrm{Pfaff}(M)$ of a given matrix M. A good reference is in [2, XV,§9]. Especially important theorem we need to know is the following lemma.

LEMMA 2.2. *(*[2, XV,Theorem 9.1]*) Let R be a commutative ring. Let $(m_{ij}) = M$ be an alternating matrix with $g_{ij} \in R$. Then*

$$\det(M) = (\mathrm{Pfaff}(M))^2.$$

Furthermore, if C is an $n \times n$ matrix in R, then

$$\mathrm{Pfaff}(CM{}^tC) = \det(C)\,\mathrm{Pfaff}(M).$$

PROOF. (of Proposition 2.1) We may assume that k is algebraically closed and that P is a k-valued point. Let us represent the tangent map of f at P by $T_P f$. One may choose a local coordinate system $x_1, \ldots, x_{2n}$ on X around P such that the symplectic form ω_X at P is represented by the matrix h when expressed in terms of $dx_1, \ldots, dx_{2n}$. Likewise one may choose a local coordinate system $y_1, \ldots, y_{2n}$ around $f(P)$ such that the symplectic form ω_Y at $f(P)$ is represented by the matrix h when expressed in terms of $dy_1, \ldots, dy_{2n}$. Then using the base $\partial/\partial x_1, \partial/\partial x_2, \ldots \partial/\partial x_{2n}$ $\partial/\partial y_1, \partial/\partial y_2, \ldots \partial/\partial y_{2n}$, $T_P f$ may be identified with a $2n \times 2n$-matrix. Since by hypothesis $T_P f$ reserves the symplectic form, we have

$$({}^tT_P f)h(T_P f) = h.$$

Let us compare the Pfaffian of the both hand sides.

$$\det(T_P f)^2 \cdot \mathrm{Pfaff}(h) = \mathrm{Pfaff}(({}^tT_P f)hT_P f) = \mathrm{Pfaff}(h).$$

Since $\mathrm{Pfaff}(h) = 1$, we conclude that the determinant of $T_P f$ should be equal to 1 or (-1). □

Note: It goes without saying that when X is connected, and if the coordinate systems $x_1, \ldots, x_{2n}$ and $y_1, \ldots, y_{2n}$ are able to chosen globally (for example if X, Y are affine space $\mathbb{A}^{2n} = \mathrm{Spec}\, k[T_1, T_2, T_3, \ldots, T_{2n}]$ with

$$\omega = dT_1 \wedge dT_{n+1} + dT_2 \wedge dT_{n+2} + dT_3 \wedge dT_{n+3} + \cdots + dT_n \wedge dT_{2n}$$

as the symplectic form), then the Jacobian of f should either be 1 on the whole of X or be -1 on the whole of X.

References

[1] T. Y. Lam, *Serre's conjecture (Lecture notes in mathematics 635)*, Springer Verlag, 1978.

[2] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.