DOLBEAULT COMPLEX OF NON-COMMUTATIVE PROJECTIVE VARIETIES.

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Contents

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Part 1. What we are talking about.

1. Introduction

In this talk we consider a non commutative version of the Kähler manifold $\mathbb{P}^n(\mathbb{C})$. We would like to consider it when the base field $\Bbbk$ is of characteristic p non zero. Well this may be quite confusing from the beginning. What is "$\mathbb{P}^n(\mathbb{C})$" over a field $\Bbbk$? In terms of (a little bit sophisticated) mathematics, we may explain our situation as follows: For the complex projective space $\mathbb{P}^n_{\mathbb{C}}$, We consider its Weil restriction $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^n_{\mathbb{C}})$ and consider its base extension $P = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^n_{\mathbb{C}}) \times_{\mathbb{R}} \mathbb{C}$. What we call "holomorphic" and "anti-holomorphic local functions on $\mathbb{P}^n(\mathbb{C})$ are understood to be a holomorphic function on P. The reader may soon realize that our P is isomorphic to the product $\mathbb{P}^n \times \mathbb{P}^n$. The space P is actually defined over $\mathbb{Z}$, so we may consider P over any base field $\Bbbk$. This is what we call "$\mathbb{P}^n(\mathbb{C})$ over a base field $\Bbbk$". You see? The idea is easy. In local terms, let us consider a set of local coordinates $z_1, \dots, z_n$ and its "complex conjugate" $\bar{z}_1, \dots \bar{z}_n$. All we need to then is to reconsider $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$ as a set of $2n$ independent variables. There is one thing we need to be careful, though: We would think that the Kähler structure of $\mathbb{P}^n(\mathbb{C})$ may then be interpreted as a holomorphic non degenerate 2-form on P. This is not true. For example, consider the case where $n = 1$. In terms of the affine coordinate z, the Kähler form is given by

$$\frac{dz d\bar{z}}{1 + z\bar{z}}.$$

The form is surely well-defined on $\mathbb{P}^1$. However, when we consider it on P, that means, when we consider z and $\bar{z}$ as independent variables, the form is not holomorphic any more. It has an obvious pole at " $1 + z\bar{z} = 0$". We need to come to this point again later.

Now, as we said, our objective here is to consider a non commutative version of $P = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^n_{\mathbb{C}}) \times_{\mathbb{R}} \mathbb{C}$. As such, Our "non commutative space" is different from what we have used in my papers [4], [5].

2. Prototype

2.1. **Marsden Weinstein quotient.** In this section let us explain a prototype of what we will do in this talk. To do so, we first review a technique of taking quotient in the world of symplectic manifold, namely, the Marsden Weinstein quotient. The entry in wikipedia:

`https://en.wikipedia.org/wiki/Moment_map`

is good enough to consult.

Let us consider the procedure of taking usual quotient

$$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1}/\mathbb{C}^\times.$$

There are some points to notice:

- The complex Lie group $\mathbb{C}^\times$ is a complexification of a compact Lie group S^1.
- The moment map of the S^1 action is given by

  $$\mathscr{m}_R = \sum_i X_i \bar{X}_i - R$$

  (where $X_0, \dots X_n$ are linear coordinates. Bar here means the usual complex conjugation.) The moment map here means that the differential 1-form $d\mathscr{m}_R$ corresponds, via the duality caused by the Kähler form $\omega = \sum_i dX_i d\bar{X}_i$, to the vector field v on $\mathbb{C}^n$ which is equal to the infinitesimal action of $\mathrm{Lie}\, S^1$. In terms of Poisson bracket, we may equally state the fact as:

$$\{\mathscr{m}_R, f\} = v.f \tag{2.1}$$

A general theory of Marsden-Weinstein quotient then tells us:

1. The fiber $\mathscr{m}_R^{-1}(0)$ of the moment map is invariant under the action of the original Lie group S^1. In our case we notice that $\mathscr{m}_R^{-1}(0)$ is equal to the sphere of radius $\sqrt{R}$ (with the origin as the center) in $\mathbb{C}^{n+1}$.
2. $\mathscr{m}_R^{-1}(0)$ is (for general R) isomorphic to $\mathbb{C}^{n+1}/\mathbb{C}^\times$. (In our case, R is enough "general" if $R > 0$.)

2.2. **Non-commutative case.** Let us consider the non-commutative version of the Marsden Weinstein quotient. (Here we said "the version". We shall use the word "version" throughout this talk. This should sound like we think the symplectic or commutative world is the "real" thing. But the author's opinion is actually the opposite: non-commutative theory is nearer to the "real" thing and the symplectic theory is just a shadow of the non-commutative theory. We are studying the real thing by its shadow. (;-)

First of all, we need to consider the non-commutative version of the affine space $\mathbb{C}^{n+1}$. We do this by specifying the "function algebra" on it. The function algebra is the Weyl algebra

$$\mathrm{Weyl}_{n+1} = \Bbbk[X_0, \dots, X_n, \bar{X}_0, \dots, \bar{X}_n].$$

Here $X_0, \dots, X_n, \bar{X}_0, \dots, \bar{X}_n$ are $2n+2$ independent variables subject to the following "canonical commutation relations(CCR)":

$$\begin{aligned} &[\bar{X}_i, X_j] = \delta_{ij} \quad \text{(Kronecker's delta)}, \\ &[\bar{X}_i, \bar{X}_i] = 0, \\ &[X_i X_j] = 0. \qquad (i, j = 0, 1, 2, \dots, n). \end{aligned}$$

Secondly, Let us consider the $\mathbb{G}_m$-action on the Weyl algebra. The action of $c \in \mathbb{G}_m$ is given as follows:

$$X_i \mapsto cX_i \bar{X}_i \mapsto c^{-1}\bar{X}_i, E_i \mapsto cE_i, \bar{E}_i \mapsto c^{-1}\bar{E}_i.$$

The infinitesimal version of the action above is given by a derivation

$$D : X_i \mapsto X_i, \quad \bar{X}_i \mapsto -\bar{X}_i,$$

This is equal to the derivation $\mathrm{ad}(\mu_R) = \mathrm{ad}(\sum_i X_i \bar{X}_i - R)$ (where R is an arbitrary constant.). This says that the moment map is given by the element μ_R. Why? Well look at the resemblance of the commutation relation

$$[\mu_R, f] = Df$$

and the Poisson bracket formula (2.1). Yes, the famous prescription of "substitute Poisson bracket to commutators" works.

We proceed to consider a non-commutative version of "$\mathbb{C}^{n+1}/S^1$." This is an easy task. The function algebra of the non commutative version is given by the invariant ring

$$(\mathrm{Weyl}_{n+1})^{S^1} = (\mathrm{Weyl}_{n+1})_{(0)}$$

where $(\mathrm{Weyl}_{n+1})_{(0)}$ is the degree 0 part of the Weyl algebra when we introduce Weyl with the grading $X_i \mapsto 1, \quad \bar{X}_i \mapsto -1$. Finally, we need to consider non commutative version of the sub-manifold where the moment map is equal to zero. This is done by usual technique as in usual algebraic geometry: By taking the residue ring. The function ring is given by:

$$A_{\text{prototype}} = (\text{Weyl}_{n+1})_{(0)}/(\mathcal{m}_R).$$

There is an explanation of this object(in Japanese) by the Author: `http://www.math.kochi-u.ac.jp/docky/bourdoki/erq3.dvi`

2.3. **A digest of the structure theory of $A_{\text{prototype}}$.** We point out a few things. Recall that for any $R \in \mathbb{C}$, there exists a sheaf $\mathfrak{D}_R$ of "twisted differential operators" on $\mathbb{P}^n_{\mathbb{C}}$. It is equal to the sheaf of differential operators on the "Serre twisting sheaf" $\mathcal{O}(R)$ when R is an integer. (See for example [2],[1].) When R is not an integer, there is no such thing as $\mathcal{O}(R)$, but the sheaf D_R still exists.

For the sake of simplicity, we shall treat here the case where R is not an integer. Then:

(1) There is an category equivalence between the category of $A_{\text{prototype}}$-modules and the category of $\mathfrak{D}_R$-modules.
(2) In short, the category ($\mathfrak{D}_R$-modules) can be interpreted as a category of modules over a single algebra $A_{\text{prototype}}$. This phenomenon is related to the "D-affineness" of $\mathbb{P}^n$.

In this way we may obtain a rough idea of the representation theory of $A_{\text{prototype}}$

2.4. **Use of characteristic p.** To obtain somewhat geometric information about the algebra $A_{\text{prototype}}$, it is convenient to consider it over the base ring $\Bbbk$ of characteristic p rather than the original idea where $\Bbbk$ is $\mathbb{R}$ or $\mathbb{C}$. Because when $\text{char}(\Bbbk) \neq 0$, the algebra A is finite over its center and thus may be analyzed by using the proj of the center.

2.5. **Homogenization(Use of C).** When the author has met the "D-affineness" of $\mathbb{P}^n$ 20 years ago, the author had a little bit of relaxing feeling. "We probably need no such construction as Proj. All we need is Spec." However, we shall use here the help of Proj to "complete" our object $A_{\text{prototype}}$. Namely, we add an extra variable C and homogenize the whole of the construction. Let us be more precise. We start with the homogenized Weyl algebra

$$\text{Weyl}^{(C)} = \Bbbk[X_0, \dots, X_n, \bar{X}_0, \dots, \bar{X}_n, C]$$

with the homogenized CCR

$$[\bar{X}_i, X_j] = C\delta_{ij}, \quad [X_i, X_j] = 0, \quad [\bar{X}_i, \bar{X}_j] = 0.$$

(we call it "CCRC" ((CCR with C.) Just for fun.) It is a graded algebra. The grading is given by:

$$\deg(X_i) = 1, \deg(\bar{X}_i) = 1, \deg(C) = 2.$$

Incidentally, note that this is the second grading we consider. The first one, which we denote by sdeg, was the following grading given by $\mathbb{G}m$-action:

$$\mathrm{sdeg}(X_i) = 1, \mathrm{sdeg}(\bar{X}_i) = -1, \mathrm{sdeg}(C) = 0.$$

To distinguish between the two grading, we call sdeg the signed degree. In this language, the first step to take Marsden-Weinstein quotient is to consider the subalgebra of whole elements of signed degree 0. That is,

$$(\mathrm{Weyl}^{(C)})_{(0)} = \{f \in \mathrm{Weyl}^{(C)}; \mathrm{sdeg}(f) = 0\}.$$

The homogeneous moment map (or, we should rather call, "the moment element") is:

$$\mathcal{m}_R^{(C)} = \sum_i X_i \bar{X}_i - RC$$

where R is a "square radius", an element of $\Bbbk$.

We end up with the homogenized version $A^{(C)}_{\text{prototype}}$ of $A_{\text{prototype}}$:

$$A^{(C)}_{\text{prototype}} = (\mathrm{WC}^{(C)})_{(0)}/(\mathcal{m}_R).$$

When the characteristic of $\Bbbk$ is non-zero, the algebra is finite over its central subalgebra (say, Z), and the $\mathrm{Proj}(Z)$ is isomorphic to $\mathbb{P}^n \times \mathbb{P}^n$.

3. What is Deligne-Illusie theory.

There are a lot of good explanations on this topic. The original paper [3] is really good. There are also many good account for the theory on the net. For example, the author found the following article very interesting:

`http://math.bu.edu/people/potthars/writings/HdRSS.pdf`

So the author would like this part very short and suggest reading such articles instead of reading this section.

Now, (if you are still reading this part,) let us begin by considering the De Rham complex of an affine space $\mathbb{A}^n$:

$$(\Omega^\bullet, d) = (\Bbbk[t_1, \dots, t_n, dt_1, \dots, dt_n], d).$$

When the characteristic p of the field $\Bbbk$ is non-zero, it has the following sub-complex

$$\Omega^\bullet_{\text{sparse}} = (\Bbbk[t_1^p, \dots, t_n^p, t_1^{p-1} dt_1, \dots, t_n^{p-1} dt_n], 0).$$

We call the elements of the complex $\Omega^\bullet_{\text{sparse}}$ *sparse* because elements of the complex have very few terms.

The first important fact to note is:

Theorem 3.1. *The inclusion*

$$(\Omega_{sparse}, 0) \to (\Omega, d)$$

is a quasi isomorphism of the complexes of sheaves on $\mathbb{A}^n$. In other words, we have an isomorphism in cohomology

$$H^i(\Omega^\bullet, d) \cong H^i(\Omega^\bullet_{sparse}).$$

We then consider the De Rham complex of a general non-singular variety X by patching such local isomorphisms. We would obtain an isomorphism

$$H^i\mathbb{R}\Gamma(\Omega^\bullet, d) \cong H^i\mathbb{R}\Gamma(\Omega^\bullet_{\text{sparse}}).$$

The left hand side is equal to the De Rham cohomology $H^i_{\text{DR}}(X)$. The right hand side, which is a cohomology of complex with 0 as the derivative, is equal to the direct sum of $H^i(\Omega^\bullet)$. We thus have an isomorphism

$$H^i_{\text{DR}}(X) \cong \bigoplus_{j+k=i} H^j(\Omega^i)$$

which is the Hodge decomposition of De Rham complex. This provides a very nice account of the Hodge decomposition. The explanation here, however, is an oversimplification. An important point we should have actually needed to take care is that the definition of $\Omega^\bullet_{\text{sparse}}$ depends of the choice of the local coordinate system. So to make things work, we should have actually worked in derived category level. We should have needed to patch objects which look like the complexes $\Omega^\bullet$ and $\Omega^\bullet_{\text{sparse}}$ as above in a derived category.

When we deal with the projective space, however, we may by-pass such patch problem by using the linear coordinates: It is possible to define Ω_{sparse} globally on projective spaces.

4. Here comes the Dolbeault complex

Our objective is to define non commutative version of the Dolbeault complex and develop its theory analogous to the Deligne-Illusie theory. To this end, we use super theory (here we mean "super" as in super algebra, super Lie algebra...etc.) to define non commutative version of differential forms.

The starting point should be fairly reasonable: We consider Weyl algebra Weyl_{n+1} for the affine space $\mathbb{A}^{n+1}$ and consider $2(n+1)$ independent "1-forms" $E_0, \dots, E_n, \bar{E}_0, \dots, \bar{E}_n$. The anti commutation relation of these "1-forms" may be a little different from what you probably imagine: Although ordinary forms anti-commute, we introduce canonical anti-commutation relation (CAR) on them:

$$[\bar{E}_j, E_i]_+ = k\delta, \quad [E_j, E_i]_+ = 0, \quad [\bar{E}_j, \bar{E}_i]_+ = 0,$$

(In the "in-homogeneous description", i.e. without C.)

Here comes an extra variable k. Knowing that we can always go back to the "ordinary theory" by taking k to be 0, Let us begin by allowing the existence of k.

Later, we will find out that our k is very important. It corresponds to the "Fubini-Study Kähler form", or curvature. (which are essentially the same because $\mathbb{P}^n$ is Kähler-Einstein.)

OK. you probably know now what we will talk. In the next section we begin with the Weyl Clifford algebra, The algebra generated by $X, \bar{X}, E, \bar{E}$'s.

There is one thing we need to be careful. We have already introduced two kinds of grading, namely, the gradings determined by signed degree and degree. We need to introduce the third grading, the one defined by the form degree. Sorry for the inconvenience, but you probably know now why they are needed.

Part 2. Definitions.

5. Weyl-Clifford algebras

5.1. Weyl algebras. Let $\Bbbk$ be a commutative field. The Weyl algebra is the following algebra.

$$\mathrm{Weyl}_{n+1}^{(h,C)} = \Bbbk[h, C, X_0, X_1, \dots, X_n, \bar{X}_0, \bar{X}_1, \dots, \bar{X}_n]$$

where $X_i, \bar{X}_j$ are subject to the following canonical commutation relations (CCR):

$$\begin{aligned}
&[\bar{X}_i, X_j] = hC\delta_{ij} \quad \text{(Kronecker's delta)},\\
&[\bar{X}_i, \bar{X}_i] = 0,\\
&[X_i X_j] = 0. \qquad (i, j = 0, 1, 2, \dots, n).
\end{aligned}$$

C, h are both central.

C is a variable to homogenize the whole story. h is a "small constant" such that the limit '$h \to 0$' gives the commutative counter part of the theory.

We note that the following identity holds. It will be needed in later arguments.

$$(X_i\bar{X}_i)^p - (hC)^{p-1} X_i \bar{X}_i = X_i^p \bar{X}_i^p \qquad (i = 0, 1, 2, \dots, n).$$

5.2. **CAR(Clifford algebra).** We define the Clifford algebra as follows.

$$\mathrm{Cliff}_{n+1}^{(h,C,k)} = \Bbbk[h, C, k, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n]$$

where the generators satisfy the following canonical anti-commutation relations(CAR):

$$[\bar{E}_i, E_j]_+ = Chk\delta_{ij}$$
$$[\bar{E}_i, \bar{E}_j]_+ = 0$$
$$[E_i, E_j]_+ = 0$$

Note that

$$(E_i\bar{E}_i)^2 = khCE_i\bar{E}_i$$

holds so that we have

$$(E_i\bar{E}_i)^p = (khC)^{p-1}E_i\bar{E}_i.$$

5.3. **Weyl-Clifford algebra.** We define the Weyl-Clifford algebra as follows.

$$\mathrm{WC}_{n+1}^{(h,C,k)} = \mathrm{Weyl}_{n+1}^{(h,C)} \otimes_{\Bbbk[h,C]} \mathrm{Cliff}_{n+1}^{(h,C,k)}$$

5.4. **the degree and the signed degree.** As explained in Part 1, we introduce the degree and the signed degree on WC. They are determined as follows:

$$\deg(X_i) = 1, \deg(\bar{X}_i) = 1, \deg(E_i) = 1, \deg(\bar{E}_i) = 1, \deg(C) = 2.$$

$$\mathrm{sdeg}(X_i) = 1, \mathrm{sdeg}(\bar{X}_i) = -1, \deg(E_i) = 1, \mathrm{sdeg}(\bar{E}_i) = -1, \mathrm{sdeg}(C) = 0.$$

5.5. GL**-action.** The Weyl Clifford algebra admits a GL-action. Namely, for any element $(g_{ij}) \in \mathrm{GL}_{n+1}(\Bbbk)$, we have

$$\begin{cases} X_i \mapsto \sum_j g_{ij}X_j \\ \bar{X}_k \mapsto \sum_l \breve{g}_{kl}X_l \\ E_i \mapsto \sum_j g_{ij}E_j \\ \bar{E}_k \mapsto \sum_l \breve{g}_{kl}E_l \end{cases}$$

where $(\breve{g}_{kl})$ is the transpose of the inverse of (g_{ij}):

$$\sum_j g_{ij}\breve{g}_{kj} = \delta_{ik}.$$

6. SUPER COMMUTATOR AND SUPER ADJOINT

6.1. Form degree and the super algebra structure of WC. We define the form degree of elements of WC by

$$\mathrm{fdeg}(X_i) = 0, \mathrm{fdeg}(\bar{X}_i) = 0, \mathrm{fdeg}(E_i) = 1, \mathrm{fdeg}(\bar{E}_i) = -1,$$
$$\mathrm{fdeg}(k) = 2, \mathrm{fdeg}(h) = 0, \mathrm{fdeg}(C) = 0.$$

We employ the super algebra structure of the Weyl Clifford algebra WC defined in the preceding section by using fdeg as the super grading.

6.2. commutators and adjoints in super algebras. Let A be a super algebra. For any homogeneous elements f, g of the algebra A, we define their *super commutator* $[f, g]$ as

$$[f, g] \overset{\mathrm{def}}{=} fg - (-1)^{\hat{f}\hat{g}} gf$$

where $\hat{f}$, $\hat{g}$ are their super degree. We extend the super commutator linearly and define it for any pair of the algebra A.

For any element a of A, we define the *super adjoint* $\mathrm{ad}(a)$ as

$$\mathrm{ad}(a) : A \ni x \mapsto [a, x] \in A.$$

7. SOME IMPORTANT ELEMENTS AND OPERATORS.

7.1. GL-invariant elements $\varepsilon, \bar{\varepsilon}$. The algebra WC has specific GL-invariant elements[1]

$$\varepsilon = \sum_i \bar{X}_i E_i, \qquad \bar{\varepsilon} = \sum_i X_i \bar{E}_i$$

7.2. $\partial, \bar{\partial}$ as the GL-invariant derivations.

$$hC\partial = \mathrm{ad}\,\varepsilon, \qquad hC\bar{\partial} = -\,\mathrm{ad}\,\bar{\varepsilon}.$$

$$\partial : \begin{cases} X_i \mapsto E_i \\ \bar{X}_i \mapsto 0 \\ E_i \mapsto 0 \\ \bar{E}_i \mapsto k\bar{X}_i. \end{cases} \qquad \bar{\partial} : \begin{cases} X_i \mapsto 0 \\ \bar{X}_i \mapsto \bar{E}_i \\ E_i \mapsto -kX_i \\ \bar{E}_i \mapsto 0. \end{cases}$$

[1] In this writing, the author frequently use ε' instead of $\bar{\varepsilon}$. It was author's old notation. Since there are too many ε' 's going around, the author desided not to fix them and simply remind here that $\varepsilon' = \bar{\varepsilon}$. Sorry for that. (2016/8/25 16:27:07 JST)

8. The element m

Let us put $m = RC - \sum X_i \bar{X}_i$. It plays an important role in our calculation.

8.1. $m^{[l]}$**, the falling factorial power of** m**.** For any non-negative integer l, we denote by $m^{[l]}$ the following "generalized factorial power of m":

$$m^{[l]} = m(m - Ch)(m - 2Ch) \dots (m - (l-1)Ch).$$

8.2. **formula of** m**.** In this section, we do some calculations on m needed for our later use. The result is summarized in the following lemma.

Lemma 8.1. *We have:*
(1) $\bar{\partial} m = -\varepsilon'$.
(2) $[m, \varepsilon'] = -Ch\varepsilon'$.
(3) $m\varepsilon' = \varepsilon'(m - Ch)$.
(4) $\bar{\partial}(m^{[l]}) = -lm^{[l-1]}\varepsilon'$ $(l = 0, 1, 2, 3, \dots)$.

Proof. (1) Knowing that $m = \frac{1}{k}\sum E_i E'_i$, we have

$$\begin{aligned} \bar{\partial} m =& \frac{1}{k}\sum_i(-kX_iE'_i) \\ =& -\sum_i X_iE'_i \\ =& -\varepsilon'. \end{aligned}$$

□

(2):

$$\begin{aligned} [m, \varepsilon'] =& \frac{1}{k}([\sum_i E_i\bar{E}_i, \varepsilon'] \\ =& -\frac{1}{k}\sum_i [E_i, \varepsilon']\bar{E}_i \\ =& -\frac{1}{k}\sum_i ChkX_i\bar{E}_i \\ =& -Ch\varepsilon' \end{aligned}$$

(3) is a trivial consequence of (2).
(4): Induction in l. The case $l = 0$ is trivial. The case $l = 1$ is treated in (1).

$$\begin{aligned}
\bar\partial m^{[l]} =&\bar\partial(m^{[l-1]}(m-(l-1)Ch))\\
=&\bar\partial(m^{[l-1]})\cdot(m-(l-1)Ch)+m^{[l-1]}\bar\partial m && \text{(Leibniz rule)}\\
=&-(l-1)m^{[l-2]}\varepsilon'\cdot(m-(l-1)Ch)-m^{[l-1]}\varepsilon' && \text{(Induction hypothesis)}.\\
=&-(l-1)m^{[l-2]}\cdot(m-(l-2)Ch)\varepsilon'-m^{[l-1]}\varepsilon' && \big(\text{Consequence of (3)}\big).\\
=&-(l-1)m^{[l-1]}\varepsilon'-m^{[l-1]}\varepsilon' && \big(\text{by definition of } m^{[\bullet]}\big)\\
=&-lm^{[l-1]}\varepsilon'
\end{aligned}$$

For a constant $R \in \Bbbk$ we put

$$\mu_R = k\sum_i X_i\bar X_i + \sum_i E_i\bar E_i - kRC.$$

The we define

$$A^{\rm pre} = ({\rm WC})_0/(\mu_R).$$

This essentially is the non-commutative analogue of the Marsden Weinstein quotient.

We need to get rid of the torsions.

$$A = A^{\rm pre}/(k\text{-torsions})$$

8.3. **get rid of k-torsions.**

Lemma 8.2. *Let $a_0, a_1, \dots, a_n$ be mutually commuting elements of a ring R. Assume there exists a central element $c \in R$ such that $a_i^2 = ca_i$ holds for each $i = 0, 1, 2, \dots, n$. Then the sum $s = \sum_{i=0}^n a_i$ satisfies the following equation:*

$$s(s-c)(s-2c)\dots(s-(n+1)c) = 0$$

Proof. We may assume that c is transcendent over $\mathbb{Q}$ and that $R = K[a_0, a_1, \dots, a_n]$ with $K = \mathbb{Q}(c)$. (Otherwise we may use a specialization argument.) In that case, we have

$$\operatorname{Spec}(R) = \prod_i \operatorname{Spec} K[a_i] \cong \{0, c\}^{n+1}.$$

The lemma now is an easy calculation of functions on the finite set $\{0, c\}^{n+1}$. □

Corollary 8.3. *Let us consider the ring $A^{\rm pre}$ and let us put $m = RC - \sum X_i\bar X_i$. Then,*

$$k^{n+2}m^{[n+2]} = k^{n+2}m(m-Ch)\dots(m-(n+1)Ch) = 0.$$

Proof.
$$(E_i\bar{E}_i)^2 = Ckh(E_i\bar{E}_i).$$
□

Corollary 8.4. *Let us put $m = RC - \sum X_i\bar{X}_i$. Then the equation $m^{[n+2]} = 0$ holds in A.*

In short, we admit the expression like
$$m = \frac{1}{k}\sum E_i\bar{E}_i.$$
In what follows, we will see that this is the only thing to note when we pass from A^{pre} to A.

Lemma 8.5. *Let n be a positive integer. Let l be a non negative integer. We assume that we are given idempotent elements $p_0, p_1, \ldots, p_n$ which are mutually commutative. We put $S = \sum_{j=0}^n p_j$. Then we have:*

$$(**) \qquad S(S-1)(S-2)\ldots(S-(l-1)) = l! \sum_{i_1<i_2<\cdots<i_l} p_{i_1}p_{i_2}\ldots p_{i_l}.$$

In particular, if $l!$ is invertible in $\Bbbk$, then we have

$$(*) \qquad \frac{1}{l!}S(S-1)(S-2)\ldots(S-(l-1)) = \sum_{i_1<i_2<\cdots<i_l} p_{i_1}p_{i_2}\ldots p_{i_l}.$$

Proof. Let us first prove (*) when the characteristic of the field $\Bbbk$ is 0. We regard the both sides of the equation (*) as functions on $2^{\{0,1,\ldots,n\}}$. In other words, we regard them as (complex-valued) measure over $X = \{0, 1, 2, \ldots, n\}$. Each p_i is then the delta measure concentrated at i.

Now for each subset A of X, the value (measure) of the right hand side at A is the number of subsets $\{i_1, i_2, \ldots, i_l\}$ of order l in A. It is equal to the combination of l objects from A, that is,
$$\binom{\#A}{l} = \frac{\#A(\#A-1)(\#A-2)\ldots(\#A-l-1)}{l!}.$$
This is equal to the value of the left hand side of (*) at A.

Let us note that the equation (**) is true when we consider it over the base ring $\mathbb{Z}$. Then by a specialization argument, we see that the equation (**) is valid for any ring. □

9. SUPPLEMENT

We have shown that $m^{[n+2]}$ is a k-torsion in A^{pre} so that it is equal to zero in A. As a supplement, in this section we show that $m^{[n+2]}$ is not zero in A^{pre}. This section is not essential for the understanding of the present paper and may be skipped.

Let us consider a $\mathbb{k}$-algebra homomorphism φ from WC_{n+1} to $\mathrm{Weyl}_{n+1}^{(0,C)}$ which sends each of the elements $k, E, \bar{E}, h$ to 0. Then we see that $\varphi(\mu_R) = 0$ so that φ (restricted to $(\mathrm{WC})_0$) descends to a $\mathbb{k}$-algebra homomorphism $\tilde{\varphi} : A^{\mathrm{pre}} \to \mathrm{Weyl}_{n+1}^{(0,C)}$. We note that $\mathrm{Weyl}_{n+1}^{(0,C)}$ is isomorphic to a usual (commutative) polynomial algebra in $X, \bar{X}, C$ variables and therefore we see that $\tilde{\varphi}(m^{[n+1]}) = (RC - \sum_i X_i \bar{X}_i)^{n+1} \neq 0$ as required.

10. A_{sparse}

. We define [2]

$$\begin{aligned}\mathrm{WC}_{\mathrm{sparse}} = \mathbb{k}[h, C, & X_0, \dots, X_n, \\ & dX_0, \dots, dX_n, \\ & \bar{X}_0^p, \dots, \bar{X}_n^p, \\ & \bar{X}_0^{p-1} d\bar{X}_0, \dots, \bar{X}_n^{p-1} d\bar{X}_n, \\ & \sum_j \bar{X}_j dX_j]\end{aligned}$$

We define $(\mathrm{WC}_{\mathrm{sparse}})_0$ to be equal to the intersection of $\mathrm{WC}_{\mathrm{sparse}}$ with WC_0.

Let us recall that we have defined our algebra A^{pre} and A as quotients of WC_0. We define $A_{\mathrm{sparse}}^{\mathrm{pre}}$ (respectively, A_{sparse}) as the image of $(\mathrm{WC}_{\mathrm{sparse}})_0$ in A^{pre} (respectively, the image of $(\mathrm{WC}_{\mathrm{sparse}})_0$ in A).

11. Statement of the main theorem

We now state our main theorem of this talk:

Theorem 11.1. *The inclusion*

$$(A_{\mathrm{sparse}}, 0) \hookrightarrow (A, \bar{\partial})$$

gives a quasi isomorphism

$$(\mathcal{A}_{\mathrm{sparse}}, 0) \hookrightarrow (\mathcal{A}, \bar{\partial})$$

of sheaves over $\mathbb{P}^n \times \mathbb{P}^n$.

[2] The author forgot to drop off k. k is actually a coboundary (locally) as we will see later. (This correction is made on Fri Aug 19 09:27:44 JST 2016.)

12. LOCAL TERMS

It would be important to use local coordinates and describe the situation locally. Because that way one may understand the algebras more clearly.

Let us consider an open set $U^\heartsuit$ of $\mathbb{P}^n \times \mathbb{P}^n$ where $X_0 \neq 0$. Let us denote by $\bullet^\heartsuit$ the "localization" of our objects to $U^\heartsuit$. To be more accurate, we consider the global sections $\Gamma(U^\heartsuit, \bullet\ )$ of sheafification $\tilde{\bullet}$ of the object $\bullet$. Let us begin by the Weyl Clifford algebra:

$$\mathrm{WC}^\heartsuit = \mathrm{WC}[X_0^{-1}].$$

It has the 0-part:

$$(\mathrm{WC})_0^\heartsuit = \Bbbk[k, h, C, x_0, \dots, x_n, x_0', \dots, x_n', e_0, \dots, e_n, e_0', \dots, e_n']$$

where we put

$$x_i = X_i X_0^{-1}, \quad x_i' = X_0 \bar{X}_i, \quad e_i = E_i X_0^{-1}, \quad e_i' = X_0 \bar{E}_i.$$

Note that we have $x_0 = 1$ so we can drop it off.

Let us next consider the localzation $A^\heartsuit$ of our main object A. It is a quotient of $(\mathrm{WC})_0^\heartsuit$. The main relation is given by

$$\mu_R = k \sum_i x_i x_i' + \sum e_i e_i' - RkC = 0.$$

Since we deleted k-torsions, we may as well write:

$$\frac{1}{k} \sum_i e_i e_i' = RC - \sum_i x_i x_i'.$$

Let us put the left hand side of the equation as m and rewrite the above equation as

$$x_0' = RC - \sum_{i=1}^n x_i x_i' - m.$$

Then by using this equation we may eliminate the variable x_0' and obtain the following expression of $A^\heartsuit$.

$$A^\heartsuit = \Bbbk[k, h, C, x_1, \dots, x_n, x_1', \dots, x_n', e_0, \dots, e_n, e_0', \dots, e_n', m].$$
$$(m = \frac{1}{k} \sum_{i=0}^n e_i e_i')$$

The generators which appear above satisfy the following CCR and CAR:

$$[x_i, x_j] = 0, \quad [x'_i, x'_j] = 0, \quad [x'_i, x_j] = hC\delta_{ij} \quad (i, j = 1, \dots, n)$$
$$[e_i, e_j]_+ = 0, [e'_i, e'_j]_+ = 0, [e'_i, e_j]_+ = Chk\delta_{ij} \quad (i, j = 0, 1, \dots, n).$$

In other words, $A^\heartsuit$ is an algebra obtained by adjoining an element m to an algebra $\Bbbk[k, h, C, x_1, \dots, x_n, x'_1, \dots, x'_n, e_0, \dots, e_n, e'_0, \dots, e'_n]$ which is isomorphic to the tensor product $\mathrm{Weyl}_n^{h,C} \otimes_{\Bbbk[h,C]} \mathrm{Cliff}_{n+1}^{h,C,k}$ of a Weyl algebra and a Clifford algebra. We note that this isomorphism preserves the 'anti holomorphic' derivation $\bar\partial$. and that it does not preserve the 'holomorphic' derivation ∂.

Proposition 12.1. *As an algebra, $A^\heartsuit[\frac{1}{k}]$ is isomorphic to a tensor product of a Weyl algebra and a Clifford algebra:*

$$A^\heartsuit[\frac{1}{k}] \cong (\mathrm{Weyl}_n \otimes \mathrm{Cliff}_{n+1})(\Bbbk[h, C, k, \frac{1}{k}]) \tag{12.1}$$

As an algebra with a derivation $\bar\partial$, It is isomorphic to a "Weyl-Clifford algebra with an extra varialbe e_0."

$$(A^\heartsuit[\frac{1}{k}], \bar\partial) \cong ((\mathrm{Cliff}_1, \bar\partial) \otimes (\mathrm{WC}_n, \bar\partial)) \otimes_{\Bbbk[h,C,k]} \Bbbk[h, C, k, \frac{1}{k}] \tag{12.2}$$

where the $\bar\partial$-operator of Cliff_1 *is defined as follows* [3]*:*

$$\bar\partial e_0 = -k, \quad \bar\partial e'_i = 0$$

□

Corollary 12.2. *Every element of $A^\heartsuit$ can be written as*

$$\sum c_{I,I'J,l,I',J'} x^I e^J m^{[l]} (x')^{I'} (e')^{J'}.$$

The above corollary suggests that the ring $A^\heartsuit$ is, as an $\Bbbk[h, k, C]$-module, "independent of h". That means, it is of the form $\Bbbk[h, C, h] \otimes_{\Bbbk[h,C]} M$ for some $\Bbbk[h, C]$-module M.

It follows that

Proposition 12.3. *$A^\heartsuit$ is flat over $\Bbbk[h, C, k]$.*

Part 3. Proof of the main result.

In this part we are going to prove Theorem 11.1. The proof is a little bit technical and is probably hard to read. (Sorry.) The author hopes that someday the situation will be improved; by an invention of the new good way to describe the whole story.

[3] memo: There was a mistake here when the author wrote the "official version"(Kinosaki report). (The author stated that the $\bar\partial$-operator of Cliff_1 was 0 but it was actually not, and the author knew it. Sorry for that.)

12.1. **derivation D_0.** Let us define an even derivation D_0 on WC.

$$D_0 = \frac{1}{khC}\,\mathrm{ad}(E_0\bar{E}_0)$$

D_0 descends to an even derivation on A.

$$e_0 \mapsto e_0, \quad e_0' \mapsto -e_0', \quad e_i \mapsto 0 \quad (\forall i > 0), \quad e_i' \mapsto 0 \quad (\forall i > 0).$$

13. Refining cocycles

13.1. **Representatives of cocycles.** For a given cocycle f of the cohomology group $\mathcal{H}(\mathcal{A})$, we are going to search its good representative. We are going to do it locally. So we restrict ourselves in the affine open set $U^\heartsuit$ as in the previous section and employ the algebra $A^\heartsuit$. We first note that the element k is locally a coboundary:

$$\bar{\partial}(-e_0) = k.$$

So the cohomology group $\mathcal{H}(A^\heartsuit)$ actually consists of k-torsions. (This is ironical. We have purged k-torsions from A, and its cohomology elements are all k-torsions.) So some kind of "'Koszul-complex-type argument" is possible. Indeed, let us consider the following exact sequence:

$$0 \to A^\heartsuit \overset{\times k}{\to} A^\heartsuit \to A^\heartsuit/kA^\heartsuit \to 0 \tag{13.1}$$

$k = -\bar{\partial}e_0$ is a coboundary in $(A^\heartsuit, \bar{\partial})$, so that the multiplication "$\times k$" on the cohomology group $H(A^\heartsuit, \bar{\partial})$ is equal to 0. The connecting map of the cohomology exact sequence associated to 13.1 thus gives a surjection (of cohomological order 1):

$$H(A^\heartsuit/kA^\heartsuit) \to H(A^\heartsuit).$$

Let us now observe the surjection above in the cochain level and obtain a little more information. The following Proposition is a starting point of our whole plan.

Proposition 13.1. *We have:*

(1)
$$\mathrm{Ker}(\bar{\partial} : A^\heartsuit \to A^\heartsuit) = \left\{ \frac{1}{k}\bar{\partial}(e_0 a) \,\middle|\, a \in A^\heartsuit; \bar{\partial}(e_0 a) \in kA \right\}$$

(2) *If an element a is equal to a coboundary in $A^\heartsuit/kA^\heartsuit$, then $\frac{1}{k}\bar{\partial}(e_0 a)$ is a $\bar{\partial}$-coboundary in $(A^\heartsuit, \bar{\partial})$.*

(3) *The $\bar{\partial}$-cohomology class of $\frac{1}{k}\bar{\partial}(e_0 a)$ depends only on the residue class ($a \mod kA^\heartsuit$) of a in $A^\heartsuit/kA^\heartsuit$.*

Proof. (1):

$\supset$: obvious.

$\subset$: Take $b \in \operatorname{Ker}(\bar{\partial}; A^\heartsuit \to A^\heartsuit)$. Since we have $\bar{\partial}(-e_0 b) = kb$, by setting $a = -b$, we obtain the relation $\frac{1}{k}\bar{\partial}(e_0 a) = b$ as required.

(2): Let us assume $a = \bar{\partial}b + kc$ for some $b, c \in A^\heartsuit$. Then we have

$$\frac{1}{k}\bar{\partial}(e_0 a) = \frac{1}{k}\bar{\partial}(e_0(\bar{\partial}b + kc)) = \bar{\partial}b + \bar{\partial}(e_0 c) = \bar{\partial}(b + e_0 c)$$

(3):obviously follows from (2).

□

13.2. **Refining cocycles.** We prove:

Proposition 13.2. *Assume that the characteristic p of $\Bbbk$ is larger than n. Then the $\bar{\partial}$-cocycle is of the form $\frac{1}{k}\bar{\partial}(e_0 a)$ where a is an element of*

$$\Bbbk[C, h, x_1, \dots, x_n, dx_1, \dots, dx_n, (x'_1)^p, \dots, (x'_n)^p, (x'_1)^{p-1}e'_1, \dots, (x'_n)^{p-1}e'_n, \varepsilon]$$

Proof. In the preceding subsection, we have proved Proposition 13.1 which says that the $\bar{\partial}$-cocycle is of the form $\frac{1}{k}\bar{\partial}(e_0 a)$ where $e_0 a$ is a $\bar{\partial}$-cocycle in A/kA. We look at the cocycle condition for $e_0 a$ and refine the choice of a for five times.

Refinement 1. Elimination of e_0.

Knowing that $e_0^2 = 0$, we may assume that the element a does not contain e_0. Namely, we refine a and may assume

$$a \in \Bbbk[h, C, x_1, \dots, x_n, x'_1, \dots, x'_n, e_1, \dots, e_n, e'_0, \dots, e'_n, \{m^{[l]}\}_{l=0}^{n+1}].$$

Refinement 2. Elimination of e'_0.

Let us employ the following element

$$\varepsilon' = e'_0 + \sum_{i=1}^{n} x_i e'_i$$

in $A^\heartsuit$ and eliminate e'_0. In other words, we regard a as an element of

$$\Bbbk[h, C, x_1, \dots, x_n, x'_1, \dots, x'_n, e_1, \dots, e_n, e'_1, \dots, e'_n, \varepsilon', \{m^{[l]}\}_{l=0}^{n+1}].$$

Seeing that $(\varepsilon')^2 = 0$, the element a is at most of degree 1 in ε' variable. (We do not actually re-choose a, but) this is the second step of our refinement.

Refinement 3. Elimination of ε'. [4]

[4] ~~The elimination of this part may be handled easier if we use a 'partial integration'~~

$$\epsilon' b = (\bar{\partial} m) b = \bar{\partial}(mb) - m(\bar{\partial} b).$$

This footnote is useless. We need this part. Sorry.

Let us now take a look at the following identity: for any $b \in A$, we have

$$m^{[l]}\varepsilon' b = \bar{\partial}\left(\frac{-1}{l+1}m^{[l+1]}\right)\cdot b \equiv \frac{-1}{l+1}m^{[l+1]}\bar{\partial}b \qquad \text{(modulo coboundary.)}$$

(Note that we assumed $p > n$ so that $l+1$ is invertible in $\Bbbk$.) Using this identity, we may eliminate, up to coboundary, the terms related to ε' and assume

$$a \in \Bbbk[h, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_1, \ldots, e_n, e'_1, \ldots, e'_n, \{m^{[l]}\}_{l=0}^{n+1}].$$

This is the third refinement.

Refinement 4. Elimination of $m^{[l]}$.

As the result of the preceding three refinement, in view of the commutation relations in section 8.2, we may express a in the form

$$a = \sum_{l=0}^{t} m^{[l]} f_l \qquad (f_0, \ldots, f_t \in \Bbbk[{}^{h,C,x_1,\ldots,x_n,x'_1,\ldots,x'_n,}_{e_1,\ldots,e_n,e'_1,\ldots,e'_n}]).$$

Among such expressions, we choose the one such that the degree t in the m-variable is the smallest. Let us examine the cocycle condition for $e_0 a \mod kA$:

$$\bar{\partial}(e_0 a) = \sum_{l=0}^{t} l e_0 m^{[l-1]}\varepsilon' f_l + \sum_{l=0}^{t} e_0 m^{[l]}\bar{\partial}(f_l) \qquad \text{(modulo } k.)$$

We pay attention to the coefficients of $e_0 e'_0$ in this equation. In other words, we decompose right hand side of the equation to a sum of eigen vectors of the derivation D_0 (See subsection 12.1). We then see

$$\sum_{l=0}^{t} l e_0 m^{[l-1]} f_l = 0.$$

Or, equivalently,

$$e_0 m^{[t-1]} f_t = -\sum_{l=0}^{t-1} \frac{l}{t} e_0 m^{[l-1]} f_l.$$

This tells us that $e_0 m^{[t]} f_t$ may be expressed as a sum of terms of degree lower than t in the m-variable. (Note again that our t here is invertible in $\Bbbk$ by our assumption $p > n$.) As a consequence, by the choice of t, we see that $t = 0$. That means, we may assume (using such choice)

$$a \in \Bbbk[h, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_1, \ldots, e_n, e'_1, \ldots, e'_n].$$

This is the fourth refinement.

We may simplify the cocycle further by using

$$dx_i = \partial x_i = e_i - x_i e_0, \quad \bar{\partial} x'_i = e'_i \quad (i = 1, 2 \dots, n).$$

These elements are $\bar{\partial}$-closed:

$$\bar{\partial}(dx_i) = 0, \quad \bar{\partial}(\bar{\partial} x'_i) = 0 \qquad (i = 1, 2, \dots n).$$

Since $e_0^2 = 0$, and since we are considering an element of the type $\frac{1}{k}\bar{\partial}(e_0 a)$, we do not have to care too much about the e_0 that appear in the expression of dx_i and we may assume:

$$a \in \Bbbk[h, C, dx_1, \dots, dx_n, \bar{\partial} x'_1, \dots, \bar{\partial} x'_n, x_1, \dots, x_n, x'_1, \dots, x'_n]$$

We would like to solve the cocycle equation $\bar{\partial}(e_0 a) = 0$ in $A^{\heartsuit}/kA^{\heartsuit}$. We have now come to the point where usual theory of De Rham complex is applicable. If we avoid changing the order of x, dx's and $x', \bar{\partial} x'$'s, (that means, if we employ such "normal ordering" here), the above module with the $\bar{\partial}$ as the derivation behaves much like the De Rham(Dolbeault) complex. There remains one difference though, that we have an extra equation:

$$0 = e_0 \sum_{i=0}^{n} e_i e'_i = e_0 \sum_{i=1}^{n} dx_i \bar{\partial} x'_i.$$

Knowing that $\bar{\partial}\varepsilon \equiv \sum_{i=0}^{n} e_i e'_i$ modulo $kA^{\heartsuit}$, we may now solve the cocycle equation.

We argue locally and we may assume $x'_1 \neq 0$. Then we may divide a by ϵ. We obtain:

$$a = \alpha\varepsilon + \beta$$

where α, β are elements which does not contain e_1.

$$\bar{\partial}(e_0 a) = \bar{\partial}(\alpha)\varepsilon + \bar{\partial}(\beta) \mod k.$$

This condition is equivalent to the conditions $\bar{\partial}\alpha = 0$ and $\bar{\partial}\beta = 0$. By using the Deligne Illusie theory, we may write α, β as sums of $\bar{\partial}$-closed objects and sparse elements.

Lastly we need to eliminate k, by using the fact that k is locally a coboundary:

$$\bar{\partial} e_0 = -k$$

□

Part 4. **Projective varieties**

14. Varieties

Let $\Bbbk$ be a field with an auto-morphism $\bar{\bullet} : \Bbbk \ni x \to \bar{x} \in \Bbbk$ of order 2. One such is of course the field of complex numbers $\mathbb{C}$ (with complex conjugation), We certainly expect our theory to expect Kähler geometry of complex varieties. But we are going to do so, using the theory of ultrafilters, by studying objects over a field of characteristic $p \neq 0$.

So let us assume here (as we have done in the preceding sections) that the characteristic p of the field $\mathbb{k}$ is positive. Our typical example should be the field $\mathbb{F}_{p^2}$ (with Frobenius map.)

Let V be a usual (i.e. ("not non-commutative") projective variety over the field $\mathbb{k}$. By definition V is a sub-variety of the projective space $\mathbb{P}^n(\mathbb{k})$ for some integer n. We assume p is sufficiently larger than n. Let $I = (F_1, \ldots, F_s)$ be the homogeneous defining ideal of V. Then starting with "the homogeneous non commutative Dolbeault complex"

$$WC_V = WC_{n+1}/(f_1^p, \ldots, f_s^p, \bar{F}_1^p, \ldots, \bar{F}_s^p),$$

We define the Dolbeault complex $(\mathrm{WC}_V)_{(0)}/(\mu_R)$ in the same way as we have done for projective space. Then we may easily verify that A_V defines a sheaf of algebras $\mathcal{A}_V$ over $V \times \bar{V}$.

The sheaf of algebras $\mathcal{A}_V$ may differ from the one you would imagine. For example, even when we consider $h = 0$, $\mathcal{A}_V$ is Larger than the ordinary Dolbeault complex of V. F_1/F_2 is nilpotent, not zero in $\mathcal{A}_V$. There are also other non zero nilpotents. But the point is that $\mathcal{A}_V$ is a matrix bundle over the usual original Dolbeault complex and is likely to be "Morita equivalent" to the usual original Dolbeault complex. We are expecting that $\mathcal{A}_V$ has properties similar to the ordinary Dolbeault complex of V.

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