CONGRUENT ZETA FUNCTIONS. NO.6

YOSHIFUMI TSUCHIMOTO

6.1. Legendre symbol.

Definition 6.1. Let p be an odd prime. Let a be an integer which is not divisible by p. Then we define the **Legendre symbol** $\left(\frac{a}{p}\right)$ by the following formula.

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } (X^2 - a) \text{ is irreducible over } \mathbb{F}_p \\ -1 & \text{otherwise} \end{cases}$$

We further define

$$\left(\frac{a}{p}\right) = 0 \text{ if } a \in p\mathbb{Z}.$$

Lemma 6.2. *Let p be an odd prime. Then:*

(1) $\left(\frac{a}{p}\right) = a^{(p-1)/2} \mod p$

(2) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

We note in particular that $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.

Definition 6.3. Let p, ℓ be distinct odd primes. Let λ be a primitive ℓ-th root of unity in an extension field of $\mathbb{F}_p$. Then for any integer a, we define a **Gauss sum** τ_a as follows.

$$\tau_a = \sum_{t=1}^{\ell-1} \left(\frac{t}{\ell}\right) \lambda^{at}$$

τ_1 is simply denoted as τ.

Lemma 6.4. (1) $\tau_a = \left(\frac{a}{\ell}\right)\tau$.

(2) $\sum_{a=0}^{l-1} \tau_a \tau_{-a} = \ell(\ell - 1)$.

(3) $\tau^2 = (-1)^{(\ell-1)/2}\ell$ *($= \ell^*$ (say)).*

(4) $\tau^{p-1} = (\ell^*)^{(p-1)/2}$.

(5) $\tau^p = \tau_p$.

Theorem 6.5.

$\left(\frac{p}{\ell}\right) = \left(\frac{\ell^*}{p}\right)$ *(where $\ell^* = (-1)^{(\ell-1)/2}\ell$)*

$\left(\frac{-1}{\ell}\right) = (-1)^{(\ell-1)/2}$

$\left(\frac{2}{\ell}\right) = (-1)^{(\ell^2-1)/8}$