

CONGRUENT ZETA FUNCTIONS. NO.9

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plane conic

DEFINITION 9.1. Let k be a field. A **projective transformation** of $\mathbb{P}^n = \mathbb{P}^n(k)$ is a map

$$f : \mathbb{P}^n \rightarrow \mathbb{P}^n$$

which is given by a non-degenerate matrix $A \in \mathrm{GL}_{n+1}(k)$ as follows:

$$f([v]) = [A.v] \quad (v \in k^{n+1})$$

where $[v]$ is the class of $v \in k^{n+1}$ in \mathbb{P}^n .

We would like to prove the following proposition.

PROPOSITION 9.2. Let $F = F(X, Y, Z) \in \mathbb{F}_q[X, Y, Z]$ be a homogeneous polynomial of degree 2. We assume F is irreducible over $\overline{\mathbb{F}_q}$. Let us put $C = V_h(F)$. Then:

- (1) There exists at least one \mathbb{F}_q -valued point P in $V_h(F)$.
- (2) For any line L passing through P defined over \mathbb{F}_q , the intersection $L \cap C$ consists of two \mathbb{F}_q -valued points P and Q_L except for a case where L contacts C .
- (3) There exists a projective change of coordinate $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ such that $f(V_h(F)) = V_h(XY - Z^2)$.
- (4) The congruent zeta function of C is always equal to the congruent zeta function of \mathbb{P}^1 .

LEMMA 9.3. We have the following picture of \mathbb{P}^2 .

(1)

$$\mathbb{P}^2 = \mathbb{A}^2 \amalg \mathbb{P}^1.$$

That means, \mathbb{P}^2 is divided into two pieces $\{Z \neq 0\} = \mathbb{C}V_h(Z)$ and $V_h(Z)$.

(2)

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$

That means, \mathbb{P}^2 is covered by three "open sets" $\{Z \neq 0\}$, $\{Y \neq 0\}$, $\{X \neq 0\}$. Each of them is isomorphic to the plane (that is, the affine space of dimension 2).

Using the Lemma and the Proposition, we may easily compute the zeta function of a non-degenerate cubic equation

$$a_1X^2 + a_2XY + a_3Y^2 + b_1X + b_2Y + c$$

in \mathbb{A}^2 . (See the exercise below.)

EXERCISE 9.1. Let p be a prime. Compute the congruent zeta functions of the following two equations (varieties) over \mathbb{F}_p .

- (1) $V(X^2 + Y^2 - 1) \subset \mathbb{A}^2$.
- (2) $V(1 + Y^2) \subset \mathbb{A}^1$.
- (3) $V_h(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$.

Is there any relation between them? (Why?)