$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.05: $\boxed{\mathbb{Z}_p \text{ as a local ring.}}$

In this lecture, rings are assumed to be unital, associative and commutative unless otherwise specified.

DEFINITION 5.1. A (unital commutative) ring A is said to be a **local ring** if it has only one maximal ideal.

LEMMA 5.2. *Let A be a ring. Then the following conditions are equivalent:*

(1) *A is a local ring.*
(2) *$A \setminus A^\times$ forms an ideal of A.*

PROPOSITION 5.3. *$\mathbb{Z}_p$ is a local ring. Its maximal ideal is equal to $p\mathbb{Z}_p$.*

We may do some "analysis" such as Newton's method to obtain some solution to algebraic equations.

Newton's method for approximating a solution of algebraic equation.
Let us solve an equation

$$x^2 = 2$$

in $\mathbb{Z}_7$. We first note that

$$3^2 \equiv 2 \quad (7)$$

hold. So let us put $x_0 = 3 = [0.3]_7$ as the first approximation of the solution. The Newton method tells us that for an approximation x of the equation $x^2 = 2$, a number x' calculated as

$$x' = \frac{1}{2}(x + \frac{2}{x})$$

gives a better approximation.

$$x_0' = \frac{1}{2}([0.3]_7 + [0.3\dot{2}]_7 = [0.3\dot{1}]_7$$

So $[0.3\dot{1}]_7$ is a better approximation of the solution. In order to make the calculation easier, let us choose $x_1 = [0.31]_7$ (insted of x_0') as a second approximation.

$$x_1' = \frac{1}{2}([0.31]_7 + 2/[0.31]_7) = \frac{1}{2}([0.31]_7 + [0.3\dot{1}45\dot{2}]_7) \fallingdotseq [0.312]_7$$

We choose $x_2 = [0.312]_7$ as a second approximation.

$$x_2' = \frac{1}{2}([0.312]_7 + 2/[0.3\dot{1}2534066\dot{2}]_7) \fallingdotseq [0.31261]_7$$

We choose $x_3 = [0.31261]_7$ as a third approximation.

$$x_3' == \frac{1}{2}([0.31261]_7 + [0.3126142465066\ldots]_7) \fallingdotseq [0.312612124...]_7$$

We choose $x_4 = [0.312612124]_7$ as a third approximation.

$$x_4' = \frac{1}{2}([0.312612124]_7 + [0.3126121245652204226622131335351\ldots]_7)$$
$$\fallingdotseq [0.3126121246621102]_7$$

Exercise 5.1. Compute $[0.5]_7/[0.11]_7$

Exercise 5.2. Find a solution to

$$x^3 \equiv 5 \pmod{11^5}$$

such that $x \equiv 3 \pmod{11}$.