CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

YOSHIFUMI TSUCHIMOTO

Let us denote by $C^\infty(M,N)$ the set of all differentiable maps from M to N. A so-called "de Rham cohomology" of S^1 is computed as a cohomology of a complex

$$C^\infty(S^1;\mathbb{R}) \overset{d/dt}{\to} C^\infty(S^1;\mathbb{R}).$$

We see that:

$$H^0_{\text{de Rham}}(S^1;\mathbb{R}) = \mathbb{R}, \quad H^1_{\text{de Rham}}(S^1;\mathbb{R}) = \mathbb{R}$$

Actually, the dimension of the 0-th cohomology is related to a number of the connected component of S^1. The dimension of the 1-st cohomology is related to a number of the 'hole' of S^1. Cohomology is then a good tool to obtain numbers ("invariants") of geometric objects.

Cohomology also arises as "obstructions". Indeed, the de Rham cohomology of the S^1 tells us a hint about "which functions are integrable", etc.

In this talk we give a definition and explain some basic properties of cohomologies. But before that, we first deal with some category theory.

Definition 1.1. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\mathrm{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set
$$\mathrm{Hom}_{\mathcal{C}}(X,Y)$$
of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map("composition (rule)")
$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

satisfying the following axioms

(1) $\mathrm{Hom}(X,Y) \cap \mathrm{Hom}(Z,W) = \emptyset$ unless $(X,Y) = (Z,W)$.
(2) (Existence of an identity) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ such that
$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$
holds for any $f \in \mathrm{Hom}(S,X), g \in \mathrm{Hom}(X,T)$ $(\forall S, T \in \mathrm{Ob}(\mathcal{C}))$.
(3) (Associativity) For any objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and for any morphisms $f \in \mathrm{Hom}(X,Y), g \in \mathrm{Hom}(Y,Z), h \in \mathrm{Hom}(Z,W)$, we have
$$(f \circ g) \circ h = f \circ (g \circ h).$$

Definition 1.2. A **universe** U is a nonempty set satisfying the following axioms:

(1) If $x \in U$ and $y \in x$, then $y \in U$.
(2) If $x, y \in U$, then $\{x, y\} \in U$.
(3) If $x \in U$, then the power set $2^x \in U$.
(4) If $\{x_i | i \in I\}$ is a family of elements of U indexed by an element $I \in U$, then $\cup_{i \in I} x_i \in U$.

LEMMA 1.3. *Let U be an universe. Then the following statements hold.*

(1) *If $x \in U$, then $\{x\} \in U$.*
(2) *If x is a subset of $y \in U$, then $x \in U$.*
(3) *If $x, y \in U$, then the ordered pair $(x, y) = \{\{x, y\}, x\}$ is in U .*
(4) *If $x, y \in U$, then $x \cup y$ and $x \times y$ are in U.*
(5) *If $\{x_i | i \in I\}$ is a family of elements of U indexed by an element $I \in U$, then we have $\prod_{i \in I} x_i \in U$.*

In this text we always assume the following.

For any set S, there always exists a universe U such that $S \in U$.

EXERCISE 1.1. Let us put

$$C^\infty(\mathbb{R}; \mathbb{R}) = \{f :\in C^\infty(\mathbb{R}; \mathbb{R}), \text{support of } f \text{ is compact}\}$$

Then:

(1) Compute the cohomology group of the following complex.
$$C^\infty(\mathbb{R}; \mathbb{R}) \to C^\infty(\mathbb{R}; \mathbb{R})$$
(2) Compute the cohomology group of the following complex.
$$C^\infty_{\text{cpt}}(\mathbb{R}; \mathbb{R}) \to C^\infty_{\text{cpt}}(\mathbb{R}; \mathbb{R})$$