CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

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Generalities on categories and definition of abelian categories

Our treatment here is a (rather strange) mixture of [2],[1]

Definition 3.1. Let F, G be two functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A morphism of functros from F to G is a family of morphisms in $\mathcal{D}$:

$$f(X) : F(X) \to G(X)$$

one for each $X \in \mathrm{Ob}(\mathcal{C})$, satisfying the following condition: for any morphism $\varphi : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{f(X)} & G(X) \\ {\scriptstyle F(\varphi)}\downarrow & & {\scriptstyle G(\varphi)}\downarrow \\ F(X) & \xrightarrow{f(X)} & G(Y) \end{array}$$

is commutative.

Definition 3.2. Let $\mathcal{C}$ be a category, X, Y be objects of $\mathcal{C}$. Then an morphism $a \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an **isomorphism** in $\mathcal{C}$ if there exists $b \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that the relations

$$ab = 1_Y \qquad ba = 1_X$$

hold. Objects X, Y in a category $\mathcal{C}$ are said to be **isomorphic** if there exists at least one isomorphism between them.

Note that by combining the above two definitions, we obtain a definition of a notion of isomorphisms of functors.

Definition 3.3. A functior $F : \mathcal{C} \to \mathcal{D}$ is said to be an **equivalene of category** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that the functor GF is isomorphic to $\mathrm{Id}_{\mathcal{C}}$, and the functor FG is isomorphic to $\mathrm{Id}_{\mathcal{D}}$. If such a thing exists, we say that the two categories are **equivalent**.

Definition 3.4. Let $\mathcal{C}$ be a category. Then:

(1) s: initial $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall a \in \mathrm{Ob}(\mathcal{C}) \quad (\# \mathrm{Hom}_{\mathcal{C}}(s, a) = 1))$.
(2) t: terminal $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall a \in \mathrm{Ob}(\mathcal{C}) \quad (\# \mathrm{Hom}_{\mathcal{C}}(a, t) = 1))$.
(3) n: null $\overset{\text{def}}{\Longleftrightarrow}$ (n: initial and n:terminal)

Definition 3.5. An category $\mathcal{C}$ is an **additive category** if it satisfies the following axioms:

(A1) Any set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an additive group. The composition of morphisms is bi-additive.
(A2) There exists a null object $0 \in \mathrm{Ob}(\mathcal{C})$.
(A3) For any objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, there exists a **biproduct** of X, Y. Namely, there exists a diagram

$$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} Z \underset{i_2}{\overset{p_2}{\leftrightarrows}} Y$$

in $\mathcal{C}$ such that

$$p_1 i_1 = 1_X, \quad p_2 i_2 = 1_Y, \quad i_1 p_1 + i_2 p_2 = 1_Z$$

holds.

DEFINITION 3.6. Let $\mathcal{C}$ be a category, $X, Y \in \mathrm{Ob}(\mathcal{C})$, adn $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. An **equalizer** k of f, g is an arrow $K \to X$ in $\mathcal{C}$ which satisfies the following properties:

(1) $f \circ k = g \circ k$.
(2) k is "universal" amoung morphisms which satisfies (1). In other words, if $m : M \to X$ is a morphism in $\mathcal{C}$ such that $f \circ m = g \circ m$, then there exists a unique arrow $h : M \to K$ in $\mathcal{C}$ which satisfy

$$m = k \circ h.$$

By reversing the directions of arrows above, one may define the notion of **coequalizers**

DEFINITION 3.7. Let $\mathcal{C}$ be an additive category. Then the equalizer (respectively, coequalizer) of an arrow $f : X \to Y$ and $0 : X \to Y$ is called the **kernel** (respectively, **cokernel**) of f.

DEFINITION 3.8. An additive category $\mathcal{C}$ is said to be **abelian** if it satisfies the following axioms.

(A4-1) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a kernel $\ker(f) : \mathrm{Ker}(f) \to X$.
(A4-2) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a cokernel $\mathrm{coker}(f) : Y \to \mathrm{Coker}(f)$.
(A4-3) For any given morphism $f : X \to Y$, we have a suitably defined isomorphism

$$l : \mathrm{Coker}(\ker(f)) \cong \mathrm{Ker}(\mathrm{coker}(f))$$

in $\mathcal{C}$. More precisely, l is a morphism which is defined by the following relations:

$$\ker(\mathrm{coker}(f)) \circ \overline{f} = f \ (\exists \overline{f}), \quad \overline{f} = l \circ \mathrm{coker}(\ker(f)).$$

REFERENCES

[1] S. I. Gelfand and Y. Manin, *Methods of homological algebra*, Springer-Verlag, 1997.
[2] S. S. Mac Lane, *Categories for the working mathematicians*, Springer Verlag, 1971.