CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

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Injective and projective objects

DEFINITION 5.1. (1) A morphism $f : X \to Y$ in a category is said to be **monic** if for any object Z of $\mathcal{C}$ and for any morphism $g_1, g_2 : Z \to X$, we have

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

(2) A morphism $f : X \to Y$ in a category is said to be **epic** if for any object Z of $\mathcal{C}$ and for any morphism $g_1, g_2 : Y \to Z$, we have

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2$$

PROPOSITION 5.2. *Let $\mathcal{C}$ be an* <u>*abelian category*</u>*. Then for any morphism f in $\mathcal{C}$, we have:*

(1) *f:monic* $\iff \mathrm{Ker}(f) = 0$.
(2) *f:epic* $\iff \mathrm{Coker}(f) = 0$.

DEFINITION 5.3. Let $\mathcal{C}$ be an abelian category.

(1) An object I in $\mathcal{C}$ is said to be **injective** if it satisfies the following condition: For any morphism $f : M \to I$ and for any monic morphism $\iota : N \to M$, f "extends" to a morphism $\hat{f} : M \to I$.

$$\begin{array}{ccc} M & \xrightarrow{\hat{f}} & I \\ {\scriptstyle\iota}\uparrow & & \| \\ N & \xrightarrow{f} & I \end{array}$$

(2) An object P in $\mathcal{C}$ is said to be **projective** if it satisfies the following condition: For any morphism $f : P \to N$ and for any epic morphism $\pi : M \to N$, f "lifts" to a morphism $\hat{f} : M \to I$.

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & M \\ \| & & \downarrow{\scriptstyle\pi} \\ P & \xrightarrow{f} & N \end{array}$$

LEMMA 5.4. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

(1) *M is a direct summand of free modules.*
(2) *M is projective*

COROLLARY 5.5. *For any ring R, the category* $(R\text{-modules})$ *of R-modules* **have enough projectives.** *That means, for any object $M \in$* $(R\text{-modules})$*, there exists a projective object P and an epic morphism $f : P \to M$.*

Definition 5.6. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.) An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \stackrel{r\times}{\to} M$$

is epic.

Lemma 5.7. *Let R be a (commutative) principal ideal domain (PID). Then an R-module I is injective if and only if it is divisible.*

Proposition 5.8. *For any (not necessarily commutative) ring R, the category* (R-modules) *of R-modules* **has enough injectives.** *That means, for any object $M \in$* (R-modules), *there exists an injective object I and an monic morphism $f : M \to I$.*