CATEGORIES, ABELIAN CATEGORIES AND COHOMOLOGIES.

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Examples of derived functors

Let $\mathcal{C}$ be an abelian category. For any object M of $\mathcal{C}$, the extension group $Ext^j_{\mathcal{C}}(M, N)$ is defined to be the derived functor of the "hom" functor

$$N \mapsto \mathrm{Hom}_{\mathcal{C}}(M, N).$$

Let G be a group. Let us consider a functor

$$F^G : M \mapsto M^G = \{m \in M; \quad g.m = m(\forall g \in G)\}$$

The functor is left-exact. The derived functor of this functor

$$H^j(G, M) = R^j F^G(M)$$

is called the j-th cohomology of G with coefficients in M. Let us consider $\mathbb{Z}$ as a G-module with trivial G-action. Then we may easily verify that

$$F^G(M) = M^G \cong \mathrm{Hom}_G(\mathbb{Z}, M).$$

Thus we have

$$H^j(G, M) = \mathrm{Ext}^j_G(\mathbb{Z}, M).$$

The extension group $\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N)$ may be calculated by using either an injective resolution of the second variable N or a projective resoltuion of the first variable M.

Example 8.1. Let us compute the extension groups $\mathrm{Ext}^j_{\mathbb{Z}}(\mathbb{Z}/36\mathbb{Z}, \mathbb{Z}/108\mathbb{Z})$.

(1) We may compute them by using an injective resolution

$$0 \to \mathbb{Z}/108\mathbb{Z} \to \mathbb{Q}/108\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of $\mathbb{Z}/108\mathbb{Z}$.

(2) We may compute them by using a free resolution

$$0 \leftarrow \mathbb{Z}/36\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 36\mathbb{Z} \leftarrow 0$$

of $\mathbb{Z}/36\mathbb{Z}$.

Exercise 8.1. Compute an extension group $\mathrm{Ext}^j(M, N)$ for modules M, N of your choice. (Please choose a non-trivial example).

To compute cohomologies of G, it is useful to use $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. For any tuples $g_0, g_1, g_2, \dots, g_t$ of G, we introduce a symbol

$$[g_0, g_1, g_2, \dots, g_t]$$

and we consider the following sequence

$(*_G)$

$$0 \leftarrow \mathbb{Z} \xleftarrow{d} \bigoplus_{g_0 \in G} \mathbb{Z} \cdot [g_0] \xleftarrow{d} \bigoplus_{g_0, g_1 \in G} \mathbb{Z} \cdot [g_0, g_1] \xleftarrow{d} \bigoplus_{g_0, g_1, g_2 \in G} \mathbb{Z} \cdot [g_0, g_1, g_2] \xleftarrow{d} \dots$$

where ϵ, d are determined by the following rules.

$d([g_0]) = 1$

$d([g_0, g_1]) = [g_1] - [g_0]$

$d([g_0, g_1, g_2]) = [g_1, g_2] - [g_0, g_2] + [g_0, g_1]$

$d([g_0, g_1, g_2, g_3]) = [g_1, g_2, g_3] - [g_0, g_2, g_3] + [g_0, g_1, g_3] - [g_0, g_1, g_2]$

$\dots$

To see that the sequence $*_G$ is acyclic, we consider a homotopy

$$h([g_0, g_1, \dots, g_t]) = [1, g_0, g_1, \dots, g_t]$$

EXERCISE 8.2. Show that $h \circ d + d \circ h = \mathrm{id}$

LEMMA 8.2. (1) *Each of the modules that appears in the sequence* $*_G$ *admits an action of* G *determined by*

$$g.[g_0, g_1, g_2, \dots, g_t] = [g \cdot g_0, g \cdot g_1, g \cdot g_2, \dots, g \cdot g_t]$$

(2)

$$C_t = \bigoplus_{g_0, g_1, g_2, \dots g_t \in G} \mathbb{Z} \cdot [g_0, g_1, g_2, \dots, g_t]$$

is $\mathbb{Z}[G]$*-free*

There are several choices for the $\mathbb{Z}[G]$-basis of C_t. One such is clearly

$$\{[1, g_1, g_2, g_3, \dots, g_t]; g_1, g_2, \dots, g_t \in G\}.$$

It is traditional (and probably useful) to use another basis

$$\{\langle g_1, g_2, g_3, \dots, g_t \rangle; g_1, g_2, \dots, g_t \in G\}.$$

where

$$\langle g_1, g_2, g_3 \dots g_t \rangle = [1,\ g_1,\ g_1 g_2,\ g_1 g_2 g_3,\ \dots,\ g_1 g_2 g_3 \dots g_t].$$

Conversely we have

$$[1, a_1, a_2, \dots, a_t] = \langle a_1, a_1^{-1} a_2, a_2^{-1} a_3, \dots, a_{t-1}^{-1} a_t \rangle.$$