COMMUTATIVE ALGEBRA

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Smoothness

Let us recall the universality of polynomial algebras.

PROPOSITION 8.1. *Let A be a commutative ring. Let B be an A-algebra. That means, we assume that there is given a specific homomorphism (called the structure homomorphism) $\iota : A \to B$. Then for any family $\{b_\lambda\}_{\lambda\in\Lambda}$ of elements of B, there exists a unique ring homomorphism*

$$\varphi : A[\{X_\lambda\}_{\lambda\in\Lambda}] \to B$$

such that $\varphi|_A = \iota$ and $\varphi(X_\lambda) = b_\lambda$ for all $\lambda \in \Lambda$.

As a corollary, we see:

PROPOSITION 8.2. *Let A be a commutative ring. Then any polynomial algebra $A[\{X_\lambda\}_{\lambda\in\Lambda}]$ is 0-smooth over A.*

LEMMA 8.3. *Let A be a ring. Let B be an A-algebra. Let I be a finitely generated ideal of B. Let us denote by $\hat{B} = \varprojlim (B/I^n)$ (respectively, $\hat{I}$) the completion of B (respectively, I) with respect to the I-adic topology. Then B is I-smooth over A if and only if $\hat{B}$ is $\hat{I}$-smooth over A.*

PROOF. $B/I^n \cong \hat{B}/\hat{I}^n$ for any n.

□

COROLLARY 8.4. *Let A be a ring. Then $A[[X_1, X_2, \dots, X_n]]$ is I-smooth over A for $I = \sum_{i=1}^n X_i A[[X_1, X_2, \dots, X_n]]$.*

Note. In general, $A[[X]]$ is not 0-smooth over A. See [1] and the literatures cited there.

THEOREM 8.5. *Let A be a ring. Let B be an A-algebra with an ideal I. If B/I is 0-smooth over A, then the sequence (which appears in Lemma 04.2)*

$$0 \to I/I^2 \to \Omega^1_{B/A} \otimes (B/I) \to \Omega^1_{B/I} \to 0$$

is split exact.

The following theorem says that the converse is true if the ring B is 0-smooth.

THEOREM 8.6. *Let A be a ring. Let B be an A-algebra with an ideal I. Assume B is 0-smooth over A. If the exact sequence*

$$0 \to I/I^2 \to \Omega^1_{B/A} \otimes (B/I) \to \Omega^1_{B/I} \to 0$$

is split exact, then B/I is 0-smooth over A.

REFERENCES

[1] H. Matsumura, *Commutative ring theory*, Cambridge university press, 1986.