COMMUTATIVE ALGEBRA

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01.Review of elementary definitions on modules.

DEFINITION 1.1. A (unital associative) **ring** is a set R equipped with two binary operations (addition ("+") and multiplication ("·")) such that the following axioms are satisfied.

(Ring-1) R is an additive group with respect to the addition.
(Ring-2) distributive law holds. Namely, we have

$$a(b+c) = ab + bc, \quad (a+b)c = ac + bc \qquad (\forall a, \forall b, \forall c \in R).$$

(Ring-3) The multiplcation is associative.
(Ring-4) R has a multiplicative unit.

In this lectuer we are mainly interested in **commutative rings**, that means, rings on which the multiplication satisfies the commutativity law.

For any ring R, we denote by 0_R (respectively, 1_R) the zero element of R (respectively, the unit element of R). Namely, 0_R and 1_R are elements of R characterized by the following rules.

- $a + 0_R = a, \quad 0_R + a = a \ \forall a \in R.$
- $a \cdot 1_R = a, \quad 1_R \cdot a = a \ \forall a \in R.$

When no confusion arises, we omit the subscript '$_R$' and write $0, 1$ instead of $0_R, 1_R$.

DEFINITION 1.2. A map $R \to S$ from a unital associative ring R to another unital associative ring S is said to be **ring homomorphism** if it satisfies the following conditions.

(Ringhom-1) $f(a+b) = f(a) + f(b)$
(Ringhom-2) $f(ab) = f(a)f(b)$
(Ringhom-3) $f(1_R) = 1_S$

Our aim is to show the following.

THEOREM 1.3. *Any regular local ring is UFD.*

DEFINITION 1.4. A commutative ring A is said to be a local ring if it has only one maximal ideal.

EXAMPLE 1.5. We give examples of local rings here.

- Any field is a local ring.
- For any commutative ring A and for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, the localization $A_\mathfrak{p}$ is a local ring with the maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

LEMMA 1.6. (1) *Let A be a local ring. Then the maximal ideal of A coincides with $A \setminus A^\times$.*
(2) *A commutative ring A is a local ring if and only if the set $A \setminus A^\times$ of non-units of A forms an ideal of A.*