COMMUTATIVE ALGEBRA

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05. Length, Hilbert function, Samuel function

LEMMA 5.1. *Let*

$$0 \to L \to M \to N \to 0$$

be an exact sequence of A-modules. Then we have

$$l(L) + l(N) = l(M).$$

DEFINITION 5.2. Let $A = \oplus_{i=0}^{\infty} A_i$ be a graded algebra. We assume

(1) $l_{A_0}(A_0) < \infty$ (Length of A_0 as an A_0 module is finite.)
(2) A is generated by homogeneous elements $x_1, x_2, \dots, x_r$ where $\deg(x_i) = d_i$.

Then for any graded finite A-module M, We define its **Hilbert series** as

$$\varphi_M(t) = \sum_{j=0}^{\infty} l_{A_0}(M_j) t^j$$

PROPOSITION 5.3. *Under the assumption of the definition above, The Hilbert series φ_M is a rational function on t. More precisely, we have*

$$\prod_{j=1}^{r} (1 - t^{d_j}) \varphi_M(t) \in \mathbb{Q}[t]$$

PROPOSITION 5.4. *If a graded algebra is generated by $x_1, x_2, \dots, x_r$ of degree 1 over a ring A_0 with $l_{A_0}(A_0) < \infty$, there exists a polynomial p_M such that*

$$l(M_k) = p_M(k) \quad (\forall k >> 0).$$

We call p_M the **Hilbert polynomial** *of M.*

COROLLARY 5.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let I be an ideal of definition (That means, there exists n_0 such that $\mathfrak{m} \supset I \supset \mathfrak{m}^{n_0}$ holds.) We put $\chi_M^I(j) = l(M/I^j)$. Then there exists a polynomial p such that $p(j) = \chi_M^I(j)$ holds for $j >> 0$.*

DEFINITION 5.6. Under the hypothesis of the Corollary above, we define the **Samuel function** of M as $\chi_M^{\mathfrak{m}}(\bullet)$.

THEOREM 5.7 (Nakayama's lemma, or NAK). *Let A be a commutative ring. Let M be an A-module. We assume that M is finitely generated (as a module) over A. That means, there exists a finite set of elements $\{m_i\}_{i=1}^{t}$ such that*

$$M = \sum_{i=1}^{t} A m_i$$

holds. If an ideal I of A satisfies

$$IM = M \quad (\text{that is, } M/IM = 0),$$

then there exists an element $c \in I$ such that

$$cm = m \qquad (\forall m \in M)$$

holds. If furthermore I is contained in $\cap_{\mathfrak{m}\in \mathrm{Spm}(A)} \mathfrak{m}$ (the Jacobson radical of A), then we have $M = 0$.

Proof. Since $IM = M$, there exists elements $b_{il} \in I$ such that

$$a_i = \sum_{l=1}^{t} b_{il} a_l \qquad (1 \leq i \leq t)$$

holds. In a matrix notation, this may be rewritten as

$$v = Bv$$

with $v =^t (m_1, \dots, m_n)$, $B = (b_{ij}) \in M_t(I)$. Using the unit matrix $1_t \in M_t(A)$ one may also write :

$$(1_t - B)v = 0.$$

Now let R be the adjugate matrix of $1_t - B$. In other words, it is a matrix which satisfies

$$R(1_t - B) = (1_t - B)R = (\det(1_t - B))1_t.$$

Then we have

$$\det(1_t - B) \cdot v = R(1_t - B)v = 0.$$

On the other hand, since $1_t - B = 1_t$ modulo I, we have $\det(1_t - B) = 1 - c$ for some $c \in I$. This c clearly satisfies

$$v = cv.$$

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