ZETA FUNCTIONS

YOSHIFUMI TSUCHIMOTO

1.1. **Formal power series.**

DEFINITION 1.1. Let A be a commutative ring. Let X be a variable. A formal power series in X over A is a formal sum

$$\sum_{i=0}^{\infty} a_i X^i \quad (a_i \in A)$$

We denote by $A[[X]]$ the ring of formal power series in X. Namely,

$$A[[X]] = \{\sum_{i=0}^{\infty} a_i X^i; a_i \in A\}.$$

For any element $f = \sum_n a_n X^n$ of $A[[X]]$, we define its **order** as follows:

$$\operatorname{ord}(f) = \inf\{n; a_n \neq 0\}.$$

Then we may define a metric on $A[[X]]$.

$$d(f,g) = \frac{1}{2^{\operatorname{ord}(f-g)}}$$

EXERCISE 1.1. Show that $(A[[X]], d)$ is a complete metric space.

EXERCISE 1.2. Show that $A[[X]]$ is a topological ring. That means, it is a topological space equipped with a ring structure and operations (the addition and the multiplication) is continuous.

1.2. **generating functions.**

A generating function is a formal power series in one indeterminate, whose coefficients encode information about a sequence of numbers $\{a_n\}$ that is indexed by the natural numbers.

1.2.1. *Ordinary generating function.*

$$G_0(\{a_n\}; X) = \sum_{n=0}^{\infty} a_n X^n$$

EXAMPLE 1.2. (Examples of ordinary generating functions)

(1) A generating function of a geometric progression:

$$\sum_{i=0}^{n} a^n X^n = \frac{1}{1-aX}.$$

(2) A generating function of an arithmetic progression:

$$\sum_{i=0}^{n} (n+1) X^n = \frac{1}{(1-X)^2}.$$

1.2.2. *Dirichlet series generating function.*

$$G_1(\{a_n\}; s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

PROPOSITION 1.3. *(Euler product expression) Assume* $\{a_n\}$ *is* **multiplicative** *in the sense that*

$$\gcd(n, m) = 1 \implies a_n a_m = a_{nm}$$

holds, Then we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p;prime} \left(\sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} \right).$$