RESOLUTIONS OF SINGULARITIES.

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03. Spec and Proj.

Definition 3.1. For any commutative ring A, we define its **spectrum** as

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A; \mathfrak{p} \text{ is a prime ideal of } A\}.$$

For any subset S of A we define

$$V(S) = V_{\mathrm{Spec}\, A}(S) = \{\mathfrak{p} \in \mathrm{Spec}\, A; \mathfrak{p} \supset S\}$$

Then we may topologize $\mathrm{Spec}(A)$ in such a way that the closed sets are sets of the form $V(S)$ for some S. Namely,

$$F : \text{closed} \iff \exists S \subset A (F = V(S))$$

We refer to the topology as the **Zariski topology**.

Exercise 3.1. Prove that Zariski topology is indeed a topolgy. That means, the collection $\{V(S)\}$ satisfies the axiom of closed sets.

Exercise 3.2. Let A be a ring. Then:

(1) Show that for any $f \in A$, $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A); f \notin \mathfrak{p}\}$ is an open set of $\mathrm{Spec}(A)$.
(2) Show that given a point $\mathfrak{p}$ of $\mathrm{Spec}(A)$ and an open set U which contains $\mathfrak{p}$, we may always find an element $f \in A$ such that $\mathfrak{p} \in D(f) \subset U$. (In other words, $\{D(f)\}$ forms an open base of the Zariski topology.

Lemma 3.2. *For any ring A, the following facts holds.*

(1) *For any subset S of A, we have*

$$V(S) = \bigcap_{s \in S} V(\{s\}).$$

(2) *For any subset S of A, let us denote by $\langle S \rangle$ the ideal of A generated by S. then we have*

$$V(S) = V(\langle S \rangle)$$

Proposition 3.3. *For any ring homomorphism $\varphi : A \to B$, we have a map*

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}(B) \ni \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(A).$$

It is continuous with respect to the Zariski topology.

Proposition 3.4. *For any ring A, the following statements hold.*

(1) *For any ideal I of A, let us denote by $\pi_I : A \to A/I$ the canonical projection. Then $\mathrm{Spec}(\pi_I)$ gives a homeomorphism between $\mathrm{Spec}(A/I)$ and $V_{\mathrm{Spec}\, A}(I)$.*
(2) *For any element s of A, let us denote by $\iota_s : A \to A[s^{-1}]$ be the canonical map. Then $\mathrm{Spec}(\iota_s)$ gives a homeomorphism between $\mathrm{Spec}(A[s^{-1}])$ and $\complement V_{\mathrm{Spec}_A}(\{s\})$.*

Definition 3.5. Let X be a topological space. A closed set F of X is said to be **reducible** if there exist closed sets F_1 and F_2 such that

$$F = F_1 \cup F_2, \quad F_1 \neq F, F_2 \neq F$$

holds. F is said to be **irreducible** if it is not reducible.

Definition 3.6. Let I be an ideal of a ring A. Then we define its **radical** to be

$$\sqrt{I} = \{x \in A; \exists N \in \mathbb{Z}_{>0} \text{ such that } x^N \in I\}.$$

Proposition 3.7. *Let A be a ring. Then;*

(1) *For any ideal I of A, we have $V(I) = V(\sqrt{I})$.*
(2) *For two ideals I, J of A, $V(I) = V(J)$ holds if and only if $\sqrt{I} = \sqrt{J}$.*
(3) *For an ideal I of A, $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.*

It is knwon that $\operatorname{Spec} A$ has a structure of "locally ringed space". A locally ringed space which locally lookes like an affine spectrum of a ring is called a scheme.

Definition 3.8. Let $S = \bigoplus_{n \in \mathbb{N}} S_n$ be a $\mathbb{N}$-graded ring. We put $S_+ = \bigoplus_{n>0} S_n$.

We define

$$\operatorname{Proj}(S) = \{\mathfrak{p} \subset S; \mathfrak{p} \text{ is a homogeneous prime ideal of S}, \mathfrak{p} \not\supset S_+\}.$$

It is known that $\operatorname{Proj}(S)$ carries a ringed space strucure on it and that it is a scheme.

Definition 3.9. Let R be a ring. Let I be an ideal of R. The scheme $\tilde{X} = \operatorname{Proj}(S)$ associated to the graded ring $S = \bigoplus_{n \in \mathbb{N}} I^n$ is called **the blowing up** of X with respect to I.