$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

YOSHIFUMI TSUCHIMOTO

Playing with “digits in base n”

You should know that every positive integer may be written in decimal notation:

$$(531)_{10} = 5 \times 10^2 + 3 \times 10^1 + 1 \times 10^0.$$

Similarly, given any integer (“base”) $b \geq 2$, we may write a number as a string of digits in base n. For example, we have

$$(531)_{10} = 1 \times 7^3 + 3 \times 7^2 + 5 \times 7 + 6 \times 1 = (1356)_7.$$

Similarly, we have

$$(531)_{10} = (1356)_7 = (1023)_8 = 1000010011_2 = (213)_{16}.$$

You may also probably know (repeating) decimal expresions of positive rational numbers.

$$(531.79)_{10} = 5 \times 10^2 + 3 \times 10^1 + 1 \times 10^0 + 7 \times 10^{-1} + 9 \times 10^{-2}.$$

$$(531.79)_{10} = (1356.\dot{5}34\dot{6})_7 = (1023.62\dot{4}365605075341217270\dot{2})_8$$

Now let us reverse the order of digits. Namely, we employ a notation like this[1]:

$$[97.135]_{10} = (531.79)_{10}$$
$$[0.135]_{10} = (531)_{10}$$
$$[123.456]_{10} = (654.321)_{10}$$
$$\ldots$$

Let us do some calculation with the above notation:

$$[0.1]_{10} + [0.9]_{10} = [0.01]_{10}$$
$$[0.1]_{10} \times [0.9]_{10} = [0.9]_{10}$$
$$[0.01]_{10} \times [0.09]_{10} = [0.009]_{10}$$

You may recognize curious rules of computations. This curious notation will lead you to a new world called “the world of addic numbers”.

Exercise 0.1. Compute

$$[0.12345]_8 + [0.75432]_8$$

with our curious notation. Then do the same computation in the usual digital notation in base 10.

[1] This is our private notation.