ALGEBRAIC GEOMETRY AND RING THEORY

YOSHIFUMI TSUCHIMOTO

4.1. **ring homomorphism and spectrum.**

Lemma 4.1. *Let A,B be two ring homomorphisms. Let*

$$\alpha : A \to B$$

be a ring homomorphism (which we always assume to be unital).

Then we have a associate map

$$\mathrm{Spec}(\alpha) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

defined by

$$\mathrm{Spec}(\alpha)(\mathfrak{p}) = \alpha^{-1}(\mathfrak{p}) \qquad (\forall \mathfrak{p} \in \mathrm{Spec}(B)).$$

The map $\mathrm{Spec}(\alpha)$ *has the following properties.*

(1)
$$\mathrm{Spec}(\alpha)(\mathfrak{p}) = \{f \in A; \rho_{\mathfrak{p}}(\alpha(f)) = 0\}$$

(2)
$$\mathrm{Spec}(\alpha)^{-1}(O_f) = O_{\alpha(f)}$$
for any $f \in A$.

(3) $\mathrm{Spec}(\alpha)$ *is continuous.*

4.2. **localization of a commutative ring.** .

Definition 4.2. Let f be an element of a commutative ring A. Then we define the localization A_f of A with respect to f as a ring defined by

$$A_f = A[X]/(Xf - 1)$$

where X is a indeterminate.

In the ring A_f, the residue class of X plays the role of the inverse of f. Therefore, we may write $A[1/f]$ instead of A_f if there is no confusion.

One may define localization in much more general situation. The reader is advised to read standard books on commutative algebras.

Lemma 4.3. *Let* f *be an element of a commutative ring* A*. Then there is a canonically defined homeomorphism between* O_f *and* $\mathrm{Spec}(A_f)$*. (It is usual to identify these two via this homeomorphism.)*

Proof. Let $i_f : A \to A_f$ be the natural homomorphism. We have already seen that we have a continuous map

$$\mathrm{Spec}(i_f) : \mathrm{Spec}(A_f) \to \mathrm{Spec}(A).$$

We need to show that it is injective, and that it gives a homeomorphism between $\mathrm{Spec}(A_f)$ and O_f.

Let us do this by considering representations.

(1) $\mathfrak{p} \in \mathrm{Spec}(A)$ corresponds to a representation $\rho_{\mathfrak{p}}$.
(2) $\mathfrak{q} \in \mathrm{Spec}(A_f)$ corresponds to a representation $\rho_{\mathfrak{q}}$.
(3) $\mathrm{Spec}(i_f)$ corresponds to a restriction map $\rho \mapsto \rho \circ i_f$.

Now, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $\rho_\mathfrak{p}$ extends to A_f if and only if the image $\rho_\mathfrak{p}(f)$ of f is invertible, that means, $\rho_\mathfrak{p}(f) \neq 0$. In such a case, the extension is unique. (We recall the fact that the inverse of an element of a field is unique.)

It is easy to prove that $\mathrm{Spec}(i_f)$ is a homeomorphism. □

Let A be a ring. Let $f \in A$. It is important to note that each element of A_f is written as a "fraction"

$$\frac{x}{f^k} \qquad (x \in A; k \in \mathbb{N}).$$

One may introduce A_f as a set of such formal fractions which satisfy ordinary computation rules. In precise, we have the following Lemma.

LEMMA 4.4. *Let A be a ring, f be its element. Let us consider the following set*

$$S = \{(x, f^k); x \in A; k \in \mathbb{N}\}.$$

We introduce on S the following equivalence law.

$$(x, f^k) \sim (y, f^l) \iff (yf^k - xf^l)f^N = 0 \qquad (\exists N \in \mathbb{N})$$

Then we may obtain a ring structure on $S/\sim$ by introducing the following sum and product.

$$(x/f^k) + (y/f^l) = (xf^l + yf^k/f^{k+l})$$
$$(x/f^k)(y/f^l) = (xy/f^{k+l})$$

where we have denoted by (x/f^k) the equivalence class of $(x, f^k) \in S$.

COROLLARY 4.5. *Let A be a ring, f be its element. Then we have $A_f = 0$ if and only if f is nilpotent.*

Likewise, for any A-module M, we may define M_f as a set of formal fractions

$$\frac{m}{f^k} \qquad (m \in M; k \in \mathbb{N}).$$

which satisfy certain computation rules.

4.2.1. *Existence of a point.*

LEMMA 4.6. *Let A be a ring. If $A \neq 0$ (which is equivalent to saying that $1_A \neq 0_A$), then we have* $\mathrm{Spec}(A) \neq \emptyset$.

PROOF. Assume $A \neq 0$. Then by Zorn's lemma we always have a maximal ideal $\mathfrak{m}$ of A. A maximal ideal is a prime ideal of A and is therefore an element of $\mathrm{Spec}(A)$. □

LEMMA 4.7. *Let A be a ring, f be its element. We have $O_f = \emptyset$ if and only if f is nilpotent.*

PROOF. We have already seen that $A_f = 0$ if and only if f is nilpotent. (Corollary 4.5). Since O_f is homeomorphic to $\mathrm{Spec}(A_f)$, we have the desired result. □