ALGEBRAIC GEOMETRY AND RING THEORY

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$\boxed{\text{projective space and projective varieties.}}$

DEFINITION 7.1. Let R be a ring. A polynomial $f(X_0, X_1, \dots, X_n) \in R[X_0, X_1, \dots, X_n]$ is said to be **homogenius** of degree d if an equality

$$f(\lambda X_0, \lambda X_1, \dots, \lambda X_n) = \lambda^d f(X_0, X_1, \dots, X_n)$$

holds as a polynomial in $n+2$ variables $X_0, X_1, X_2, \dots, X_n, \lambda$.

DEFINITION 7.2. Let k be a field.

(1) We put
$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times$$
and call it (the set of k-valued points of) the **projective space**. The class of an element $(x_0, x_1, \dots, x_n)$ in $\mathbb{P}^n(k)$ is denoted by $[x_0 : x_1 : \dots : x_n]$.

(2) Let $f_1, f_2, \dots, f_l \in k[X_0, \dots, X_n]$ be homogenious polynomials. Then we set
$$V_h(f_1, \dots, f_l) = \{[x_0 : x_1 : x_2 : \dots x_n]; f_j(x_0, x_1, x_2, \dots, x_n) = 0 \qquad (j = 1, 2, 3, \dots, l)\}.$$
and call it (the set of k-valued point of) the **projective variety** defined by $\{f_1, f_2, \dots, f_l\}$.

(Note that the condition $f_j(x) = 0$ does not depend on the choice of the representative $x \in k^{n+1}$ of $[x] \in \mathbb{P}^n(k)$.)

LEMMA 7.3. *We have the following picture of* $\mathbb{P}^2$.

(1)
$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1.$$
That means, $\mathbb{P}^2$ *is divided into two pieces* $\{Z \neq 0\} = \complement V_h(Z)$ *a nd* $V_h(Z)$.

(2)
$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$
That means, $\mathbb{P}^2$ *is covered by three "open sets"* $\{Z \neq 0\}, \{Y \neq 0\}, \{X \neq 0\}$. *Each of them is isomorphic to the plane (that is, the affine space of dimension 2).*

7.1. **proj.**

DEFINITION 7.4. An $\mathbb{N}$-graded ring S is a commutative ring with a direct sum decomposition

$$S = \bigoplus_{i \in \mathbb{N}} S_i \qquad \text{(as a module)}$$

such that $S_i S_j \subset S_{i+j} \quad (\forall i, j \in \mathbb{N})$ holds. We define its irrelevant ideal S_+ as

$$S_+ = \bigoplus_{i > 0} S_i.$$

An element f of S is said to be homogenous if it is an element of $\cup S_i$. An ideal of S is said to be homogeneous if it is generated by homogeneous elements. Homogeneous subalgebras are defined in a same way.

DEFINITION 7.5.

$$\operatorname{Proj}(S) = \{\mathfrak{p}; \mathfrak{p} \text{ is a homogeneous prime ideal of } S, \mathfrak{p} \not\supset S_+\}$$

For any homogeneous element f of S, we define a subset D_f of $\operatorname{Proj}(S)$ as

$$D_f = \{\mathfrak{p} \in \operatorname{Proj} S; \mathfrak{p} \not\in f\}.$$

$\operatorname{Proj} S$ has a topology (Zariski topology) which is defined by employing $\{D(f)\}$ as an open base.

PROPOSITION 7.6. *For any graded ring S and its homogeneous element f, $S[\frac{1}{f}]$ also carries a structure of graded ring. There is a homeomorphism*

$$D_f \sim \operatorname{Spec}(S[\frac{1}{f}])_0).$$

We may define, via these homeo altogether, a locally ringed space structure on $\operatorname{Proj}(S)$.