ALGEBRAIC GEOMETRY AND RING THEORY

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Note that a sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

of sheaves of abelian groups is exact if and only if it is exact stalkwise.

PROPOSITION 9.1. *Let*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

be an exact sequence of sheaves of abelian groups on a topological space X. Then:

(1) *For any open subset U of X, the corresponding sequence*

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

of sections is exact.

(2)

$$B(U) \to \mathcal{C}(U)$$

may not be surjective in general.

In a language of category theory, the global section function

$$\Gamma(U, \bullet) : (\text{Sheaf of abelian groups on } X) \to (\text{Ab})$$

is a left exact functior. (But not exact.) To treat it, we employ derived functors.

LEMMA 9.2. (1) *For any ring A, if an A-module I is injective, then the associated sheaf $\tilde{I}$ on $\operatorname{Spec} A$ is injective.*

(2) *The category (A-modules) has enough injectives.*

(3) *For any scheme X with an affine open covering $X = \sum_j U_j$, for any $\mathcal{O}_X$-quasi coherent sheaf $\mathcal{F}$ on X, we have:*
$\mathcal{F}$: injective $\iff$ $\mathcal{F}_{U_j}$ injective for any j.

DEFINITION 9.3.

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$$

THEOREM 9.4 (Serre). *For any affine scheme $X = \operatorname{Spec}(A)$ and for any quasi coherent $\mathcal{O}_X$-module $\mathcal{F}$ on X, we have*

$$H^i(X, \mathcal{F}) = 0 \qquad (\text{for } \forall i > 0)$$

PROPOSITION 9.5. *Let X be a scheme with an affine open covering $\mathfrak{U} = \{U_j\}$. Then for any quasi coherent $\mathcal{O}_X$-module $\mathcal{F}$ on X, The cohomology $H^i(X, \mathcal{F})$ may be computed by usin the Čech cohomology with respect to $\mathfrak{U}$.*