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NON-COMMUTATIVE PROJECTIVE SPACE AS A NON-COMMUTATIVE KÄHLER MANIFOLD

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Abstract. We give a non-commutative counterpart of the complex projective space with the Fubini metric. Some cohomologies and a spectral sequence about them are computed.

Contents

1. Introduction

Complex projective space $\mathbb{P}^n_{\mathbb{C}}$ has a Fubini Study metric and the corresponding Kähler structure therefore gives a structure of a symplectic manifold. In this paper we make its non-commutative counterpart. We also decorate it with some super odd variables, so that we may talk about "differential forms" on it. What we get is a sheaf $\mathcal{A}$ of modules over $\mathbb{P}^n \times \mathbb{P}^n$ which is a quotient of another sheaf $\mathcal{WC}$ which corresponds, in the usual manner, to the algebra WC which is obtained as a tensor product of a Weyl algebra and a Clifford algebra.

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[1]2000 *Mathematics Subject Classification.* Primary 53D50 ; Secondary 14F40
Key words and phrases. Non-commutative algebraic geometry, de Rham cohomology .

The whole story was developed according to the theory of "Marsden-Weinstein" quotient as a guide, and we found out that the sheaf $\mathcal{WC}$ is interesting in its own right, even without taking the quotient. Each of our sheaves $\mathcal{WC}$ and $\mathcal{A}$ has two odd derivations $\mathfrak{d}$ and $\bar{\mathfrak{d}}$, making itself a double complex. We have computed some cohomologies of the double complexes, and by doing so we are developing something of a non-commutative Dobleault theory. Our "non-commutative Dolbeault complex" $(\mathcal{A}, \mathfrak{d}, \bar{\mathfrak{d}})$ has something different than the usual ones, to keep ourselves well-defined in our setups. And it seems that the complex has an interesting duality property.

Let us describe the story in more detail, using a framework of algebraic geometry a little more.

First of all, we need to regard $\mathbb{P}^n_{\mathbb{C}}$ as a variety over $\mathbb{R}$. In other words, we consider the Weil scalar restriction $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^n$. In addition to that, we consider its scalar extension $\mathbb{P}^n \times \mathbb{P}^n = (\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^n \times_{\mathbb{R}} \mathbb{C})$. In short, we employ holomorphic coordinates 'z'and the anti-holomorphic coordinates '$\bar{z}$' as the coordinates of $\mathbb{P}^n \times \mathbb{P}^n$: They are complex conjugate on $\mathbb{P}^n$, but we regard them as independent variables on $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^n$.

To construct the non-commutative counter-part of $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^n$, we use the theory of Marsden-Weinstein quotient.

To make ourselves clearer, we briefly explain with a simple prototype in section 2. By taking the Marsden-Weinstein quotient of a Weyl algebra we obtain an algebra $A^{\text{proto}} = (\text{Weyl})_{(0)}/(\mu)$. A^{proto} is isomorphic to $\mathfrak{U}(\mathfrak{gl})/(\mu)$ which may be identified with a twisted differential operator on $\mathbb{P}^n$. The object had been well studied within the theory of "Localization of $\mathfrak{g}$-modules"[1].

In section 3, we are going to add two structures to this object and enrich the story.

One is the compactification.

A^{proto} is certainly "compact" in the $\mathbb{P}^n$ direction ("position-variables"), but is is not "compact" in the fibers("momentum-variables"). To compactify it we would need $\mathbb{P}^n$ for both directions, which would arise $\mathbb{P}^n \times \mathbb{P}^n$, with non commutative structure. By thinking in the case of the positive characteristic, these situations can be handled visually.

The other is adjunction of super variables. We add anti-commuting variables to the Weyl algebra and move to the Weyl Clifford algebra WC. We develop the super version of Marsden-Weinstein quotient. The bare Marsden-Weinstein quotient itself are too rough, we supplement it by using the tools of homological algebras.

In our construction, there seems to be, at first glance, some options such as choosing the right derivations $\mathfrak{d}, \bar{\mathfrak{d}}$ and the moment map. But the fact is, that considerable part of our choice is inevitable. It will be explained in section 3.

We obtain, as a result of the Marsden-Weinstein quotient, a sheaf $\mathcal{A}$ of modules over $\mathbb{P}^n \times \mathbb{P}^n$. It is a quotient of a sheaf $\mathcal{WC}$ of modules. To make non-commutativity somewhat easier to handle, we are mainly working with the case where the base field $\Bbbk$ is of characteristic $p > 0$. (The whole story would then be applied to the case of characteristic 0 by using a technique of ultra-filter limits [7],[8],[2],[3].) For instance, in characteristic $p > 0$ case, we may treat the sheaves $\mathcal{A}$ and $\mathcal{WC}$ as algebras over $\mathcal{O}^{(p)}_{\mathbb{P}^n \times \mathbb{P}^n}$.

As cohomologies are determined only from the module structures, It is possible to calculate some of them (Theorem 6.1) by using theories in the usual commutative algebraic geometry. In particular, we make full use of the Deligne-Illusie-Cartier theory. We give a small account of the result of the theory in section 5.2. In a sense, we may also regard our theory as a "non-commutative Dolbeault version " of the theory. It seems that our theory of dealing with non-commutativity by passing to positive characteristic case gets along very well with the Deligne-Illusie-Cartier theory.

On the other hand, we also pay homage to the non-commutative structure, holomorphic- and anti-holomorphic- differential $\mathfrak{d}, \bar{\mathfrak{d}}$ are chosen so that it extends to the non-commutative case as discussed in section 3. In particular a degeneracy of a special spectral sequence is observed (Corollary 6.2). As our theory has a $z \leftrightarrow \bar{z}$ -symmetry, it should be an evidence of some kind of duality happening here.

So far, we need some more work to understand general projective varieties by our strategy. We are expecting to obtain a result which corresponds, in the Hodge theory, to a something which look like an isomorphism $H^{(l,m)} \cong \bar{H}^{m,l}$.

Our method here has the advantage of being "visible". It can easily connect to an existent commutative theories and so on.

2. PROTOTYPE

2.1. **The base problem.** Since our problem is fairly complicated, in this section we describe a "prototype" of our theory, a story without much decorations.

Let $\Bbbk_1$ be a commutative ring, $h \in \Bbbk_1$.

We start with (in-homogeneous) Weyl algebra

$$\mathrm{weyl}_{n+1} = \Bbbk\langle x_0, \dots x_n, \bar{x}_0, \dots \bar{x}_n\rangle/(\mathrm{ccr})$$

(where ccr is the "canonical commutation relation) $[\bar{x}_i, x_j] = h\delta_{ij}, \quad [x_i, x_j] = 0, \quad [\bar{x}_i, \bar{x}_j] = 0.$)

Let us take a submodule $\mathrm{weyl}_{(0)}$ of weyl_{n+1} which consists of elements of signed degree 0.

$$\mathrm{weyl}_{(0)} = \Bbbk\langle x_0, \dots x_n, \bar{x}_0, \dots \bar{x}_n\rangle_{(0)} = \Bbbk\langle\{x_i\bar{x}_j; i, j \in \{0, 1, 2, \dots, n\}\}\rangle,$$

where the signed degree sdeg is defined as follows.

variable :	x	$\bar{x}$
sdeg:	1	-1

Let us cut it at "places where moment map equals 0" that is, $\sum_i x_i\bar{x}_i = R$. We get an algebra A:

$$A = \mathrm{weyl}_{(0)} / (\sum_i x_i\bar{x}_i - R)$$

The algebra is isomorphic to a quotient ring of the universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}_{n+1})$ of the Lie algebra $\mathfrak{gl}_{n+1}$. Indeed, it is easy to verify that the elements $\{x_i\bar{x}_j\}$ satisfy the commutation relation of $\mathfrak{gl}$. A is deeply related to the idea of "localization of the $\mathfrak{gl}$-modules".

When the characteristic of over base ring $\Bbbk_1$ is a positive prime $p > 0$, Then the center of our algebra A is equal to

$$\Bbbk[\{x_i^p\bar{x}_j^p; i, j = 0, \dots, n\}]/(\mathrm{relation})$$

with the relation
$$\sum_i x_i^p \bar{x}_i^p = R^p(1 - h^{p-1}).$$

Fact 2.1.1. $A = \mathrm{weyl}_{(0)} / (\sum_i x_i \bar{x}_i - R)$ defines, on an open subset of $\mathbb{P}^n \times \mathbb{P}^n$ defined by
$$\{[a_0 : a_1 : \dots a_n], [\bar{a}_0 : \bar{a}_1 : \dots \bar{a}_n]; \sum_i a_i \bar{a}_i \neq 0\},$$
a coherent sheaf of algebras $\mathcal{A}$. Each of its fibers is isomorphic to the full matrix algebra M_{p^n}.

base problem

Extend the Fact 2.1.1 and construct a non-commutative version of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^n \cong \mathbb{P}^n \times \mathbb{P}^n$. In particular, keep in mind to:

- take the "completion" of the Spec of the non-uniform Weil ring.
- make take "non-commutative version of differential form" using super variables.

2.2. **Marsden-Weinstein quotient.** The word "Marsden-Weinstein quotient" itself is borrowed from symplectic geometry. For the reader who are unfamiliar with the topic, As things in internets are not permanent, we refer to [6] for a good reference.

For a ring W, we would like to "restrict" it by its subset S. When W is commutative, it is done by considering the quotient ring $W/(W \cdot S)$ of the ring W by its ideal $W \cdot S$ generated by S.

When our ring W is non-commutative, it is possible to think of the idea generated by S, But it is not always effective: W could be a simple ring like our Weyl algebra. It is generally impossible to accurately determine two variables at the same time according to the uncertainty principle.

So we consider instead the left ideal $J = W \cdot S$, and take a quotient of the idealizer $\mathbb{I}_W(J) = \{a \in W; Ja \subset J\}$ by J: We may consider the idealizer as a set of elements which are consistent with the restriction $S = 0$. and regard the quotient $\mathbb{I}_W(J)/J$ as a "restriction of W by S".

Suppose that an algebra group G is acting on a ring W. The quotient of W with G (called Marsden-Weinstein quotient) is actually obtained with the concept of "restriction" as above. Let's show below that the Marsden-Weinstein quotient can be seen as W's "restriction" of W by moment map mu.

Let us put it in another way. We may guess, as we do in the commutative theory, that the G-orbit of $\mathrm{Spec}(W)$ corresponds to the ring of G-invariants in W:
$$\mathrm{Spec}(W)/G \text{ “=” } \mathrm{Spec}(W^G)$$
We need to consider another thing. In exchange for losing some coordinate functions (stop deciding where in the G-orbit we are), we may determine values of some elements (" momentum "). In other words: there exist elements $\{\mu\} \in W^G$ such that, the ring which we really want should look like:
$$\mathrm{Spec}(W)//G = \mu^{-1}(0) \subset \mathrm{Spec}(W^G)$$
$$W^G = \mathbb{I}_W(J), \quad J = W \cdot (\{\mu\})$$

We have in summary:

summary

- For an action of a group to a non-commutative ring we may find some moment maps.
- By using a "restriction" by moment maps we obtain a quotient space of the spectrum of the algebra.

A supplementary note on symplectic quotient.

For an action of a real Lie group G on a complex Kähler manifold X which preserves the Kähler form, we have

$$X/G_{\mathbb{C}} \cong X//G \qquad (G_{\mathbb{C}} : \text{complexification of } G).$$

We have in particular

$$\mathbb{P}^n(\mathbb{C}) \cong \mathbb{A}^{n+1}_{\mathbb{C}}/\mathbb{G}_m \cong \mathbb{A}^{n+1}//S^1 = \mu^{-1}(0)/S^1.$$

We are going to make a "non-commutative version" with extra "super variables". As we have explained above, The choice of the moment map corresponds to the choice of the action of S^1, or, in other words, the choice of $\mathbb{Z}$ -grading. Since it should match with the story of P^n, above, Two choices are possible.

(1) $\sum_i X_i\bar{X}_i$. It corresponds to the grading

x_i	$\bar{x}_i$	e_i	$\bar{e}_i$
1	-1	0	0

(2) $k\sum_i X_i\bar{X}_i + \sum_i E_i\bar{E}_i$. It corresponds to the grading

x_i	$\bar{x}_i$	e_i	$\bar{e}_i$
1	-1	1	-1

We shall compare these in section 4.2.

2.3. homogeneous Weyl-Clifford algebra.

2.3.1. *Base field* $\Bbbk$, *base ring* $\Bbbk_1$. We fix a base field $\Bbbk$ and its extension commutative ring $\Bbbk_1$. We choose a special elements $h \in \Bbbk_1$ By putting this way, we can quickly return to the commutative case by specializing h to 0. In later sections, in many case $\Bbbk$ will be a field of characteristic $p > 0$, and $\Bbbk_1$ will be a ring $\Bbbk[h, \frac{1}{1-h^{p-1}}]$. To avoid some difficult problems, we always assume p is sufficiently large. (Usually $p > 2n$, the dimension of our projective space, would be enough.)

2.3.2. *Definition of homogeneous Weyl-Clifford algebra.*

Definition 2.3.1. We define *homogeneous Weyl algebra* as follows

$$\mathrm{Weyl}^{(h,C)}_{n+1} = \Bbbk_1\langle C, X_0, X_1, \ldots, X_n, \bar{X}_0, \bar{X}_1, \ldots, \bar{X}_n\rangle$$

where the generates $X_i, \bar{X}_j$ satisfy the following canonical commutation relations(CCR):

$$[\bar{X}_i, X_j] = hC\delta_{ij} \quad \text{(Kronecker's delta)},$$
$$[\bar{X}_i, \bar{X}_i] = 0, \qquad [X_i, X_j] = 0. \qquad (i, j = 0, 1, 2, \ldots, n).$$

C is a central element.

Definition 2.3.2. We define *homogeneous Clifford algebra* as follows

$$\mathrm{Cliff}_{n+1}^{(h,C,k)} = \Bbbk_1\langle C, E_0, \dots, E_n, \bar{E}_0, \dots, \bar{E}_n\rangle$$

where the generators $E, \bar{E}$ satisfy the following anti-commutation relations (CAR):

$$[\bar{E}_i, E_j]_+ = Chk\delta_{ij}$$
$$[\bar{E}_i, \bar{E}_j]_+ = 0, \qquad [E_i, E_j]_+ = 0$$

C again is a central element.

Definition 2.3.3. For any non-negative integers n, m, we define *homogeneous Weyl-Clifford algebra* as the following tensor product.

$$\mathrm{WC}_{n+1,m+1}^{(h,C,k)} = \mathrm{Weyl}_{n+1}^{(h,C)} \otimes_{\Bbbk_1[k,C]} \mathrm{Cliff}_{m+1}^{(h,C,k)}.$$

when $n = m$, for the sake of simplicity, we write $\mathrm{WC}_{n+1}^{(h,C,k)} = \mathrm{WC}_{n+1,n+1}^{(h,C,k)}$.

By a (homogeneous version of) the well-known fact we have:

Proposition 2.3.4. Weyl_{n+1} is a free module over $\Bbbk_1[k, C]$ with a free basis

$$\{X_0^{i_0} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n} |\ i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}\}.$$

Cliff_{m+1} is a free module over $\Bbbk_1[k, C]$ with a free basis

$$\{E_0^{j_0} E_1^{j_1} E_2^{j_2} \dots E_n^{j_n} |\ j_1, \dots, j_n \in \{0, 1\}\}$$

It also follows that $\mathrm{WC}_{n,m}$ and WC_n are $\Bbbk_1[k, C]$-free modules.

2.3.3. WC *as a super algebra.* As with the algebra of the differential forms, WC_{n+1} admits a structure of a super algebra by defining $X, \bar{X}, C$ as even, and $E, \bar{E}$ as odd. Hereinafter, the symbol on WC is used as a symbol as a super algebra. For example, bracket is a super commutator and ad is a super adjoint:

$$\mathrm{ad}(x)(y) = [x, y] = xy - (-1)^{\hat{x}\hat{y}} yx$$

(where $\hat{?}$ denotes the signature of ?.)

2.3.4. *Elements ε, $\bar{\varepsilon}$ and derivations $\mathfrak{d}, \bar{\mathfrak{d}}$.* We define the elements ε, $\bar{\varepsilon}$ in WC_{n+1} as follows.

$$\varepsilon = \sum_i \bar{X}_i E_i, \quad \bar{\varepsilon} = \sum_i X_i \bar{E}_i$$

We define odd derivations $\mathfrak{d}, \bar{\mathfrak{d}}$ on WC as follows.

$$\mathfrak{d} = \frac{1}{hC} \mathrm{ad}\, \varepsilon, \quad \bar{\mathfrak{d}} = -\frac{1}{hC} \mathrm{ad}\, \bar{\varepsilon}$$

(where, as we have written in the last subsection, ad denotes the super adjoint.) These are apparently operators on $\mathrm{WC} \otimes_{\Bbbk_1[k,C]} \Bbbk_1[k, C, \frac{1}{hC}]$, and sends, generators $X_?, \bar{X}_?, C, E_?, \bar{E}_?$ to elements in WC. They also satisfy super Leibniz rule. We see therefore, by using the freeness 2.3.4, that these define operators on WC.

We note also that

$$\varepsilon = \mathfrak{d}(\sum_i X_i \bar{X}_i), \quad \bar{\varepsilon} = \bar{\mathfrak{d}}(\sum_i X_i \bar{X}_i).$$

For $x \in \mathrm{WC}$,

$$[\mathfrak{d}, \bar{\mathfrak{d}}](x) = (\mathfrak{d}\bar{\mathfrak{d}} + \bar{\mathfrak{d}}\mathfrak{d})(x) = \frac{1}{hC} \operatorname{ad} \mu_0(x) = -k \operatorname{sdeg}(x) \cdot x$$

where $\mu_0 \in \mathrm{WC}$ is defined as

$$\mu_0 = k \sum_i X_i \bar{X}_i + \sum_i E_i \bar{E}_i.$$

sdeg is "signed degree ("number of the bar-ed variables")" defined as follows.

variable:	X	$\bar{X}$	E	$\bar{E}$	C
sdeg:	1	-1	1	-1	0

3. DERIVATIONS,A MOMENT MAP, AND WHY WE SELECT THEM

3.1. **derivations $\mathfrak{d}, \bar{\mathfrak{d}}$.** Let us put some assumptions and explain why we use WC, $\mathfrak{d}, \bar{\mathfrak{d}}$ as a non-commutative version of the ring of differential forms on $\mathbb{A}^{n+1}$. Let $\Bbbk_1$ be a commutative ring, $h \in \Bbbk_1$.
(Assumption 0.) S is a super algebra over $\Bbbk_1$. For elements of S, we use standard notation of super algebra. For example, $[\bullet, \bullet]$ is a super commutator instead of commutator. hat as in $\hat{\bullet}$ denotes the parity of $\bullet$.
(Assumption 1.) The algebra S has variables $x_0, \ldots, x_n, \bar{x}_0, \ldots, \bar{x}_n$ which admits CCR.
(Assumption 2.) S admits two **odd derivations** $\mathfrak{d}, \bar{\mathfrak{d}}$. In other words, $\mathfrak{d}, \bar{\mathfrak{d}}$ are two $\Bbbk_1$ linear map from S to S which satisfy the super Leibniz rule

$$\mathfrak{d}(ab) = \mathfrak{d}(a)b + (-1)^{\hat{a}} a \mathfrak{d} b$$
$$\bar{\mathfrak{d}}(ab) = \bar{\mathfrak{d}}(a)b + (-1)^{\hat{a}} a \bar{\mathfrak{d}} b$$

(Assumption 3.) We denote $\mathfrak{d}X_0, \ldots \mathfrak{d}X_n$ as $E_0, \ldots, E_n$, and $\bar{\mathfrak{d}}X_0, \ldots \bar{\mathfrak{d}}X_n$ as $\bar{E}_0, \ldots, \bar{E}_n$. $E_0, \ldots, E_n$, $\bar{E}_0, \ldots, \bar{E}_n$ satisfy the **(ACR)**.
(Assumption 4.) (**Cauchy-Riemann**) $\bar{\mathfrak{d}} x_i = 0, \mathfrak{d} \bar{x}_i = 0$ $(i = 0, 1, 2, \ldots, n)$.
(Assumption 5.) E_i's and X_j's mutually commute. In the same way, $\bar{E}_i$ and $\bar{X}_j$ mutually commutes. In other words, Both 'variables without bars' and 'variables with bars' form super algebras which are isomorphic to the super ring of ordinary differential forms.
(Consequence 6.) X_i commutes with $\bar{E}_i$. This is verified by acting $\mathfrak{d}$ or $\bar{\mathfrak{d}}$on CCR:

$$0 = \mathfrak{d}(h\delta_{ij}) = \mathfrak{d}[X_i, \bar{X}_j] = [E_i, \bar{X}_j]$$

and so on
(Consequence 7.) For any i, j, we have

$$0 = \mathfrak{d}[x_i, \bar{e}_j] = [e_i, \bar{e}_j] + [x_i, \mathfrak{d}\bar{e}_j].$$

We have therefore,

$$[x_i, \mathfrak{d}\bar{e}_j] = -\delta_{ij} k_1.$$

In particular, we have $\mathfrak{d}\bar{e}_j (= \mathfrak{d}\bar{\mathfrak{d}} x_j) \neq 0$.
(Assumption 8.) There exists a central constant k such that, $k_1 = hk$, $\mathfrak{d}\bar{\mathfrak{d}} \bar{x}_i = k \bar{x}_i$ $(\forall i)$.
(Consequence 8.)Both $\mathfrak{d}, \bar{\mathfrak{d}}$ are inner inner up to constant multiples:

$$\mathfrak{d} = \frac{1}{h} \operatorname{ad}(\sum_i \bar{x}_i e_i), \qquad \bar{\mathfrak{d}} = -\frac{1}{h} \operatorname{ad}(\sum_i x_i \bar{e}_i)$$

(Consequence 9.)

$$[\bar{\mathfrak{d}}, \mathfrak{d}] = \frac{1}{h^2}\operatorname{ad}([\sum_i \bar{x}_i e_i, \sum_j x_j \bar{e}_j) = \frac{1}{h}\operatorname{ad}(\mathfrak{d}(\sum_j x_j \bar{e}_j)) = \frac{1}{h}\operatorname{ad}(k\sum_j x_j \bar{x}_j + \sum_j e_j \bar{e}_j)$$

Conclusion

As a non-commutative counter part of the ring of differential forms on $\mathbb{A}^{n+1}$, We take the homogeneous Weyl Clifford algebra WC_{n+1}. As analogues of holomorphic- and anti-holomorphic-derivations, we adopt

$$\mathfrak{d} = \frac{1}{h}\operatorname{ad}(\sum_i \bar{x}_i e_i), \qquad \bar{\mathfrak{d}} = -\frac{1}{h}\operatorname{ad}(\sum_i x_i \bar{e}_i).$$

(Remark)

$$\sum_i \bar{x}_i e_i = \mathfrak{d}(\sum_i x_i \bar{x}_i), \qquad \sum_i x_i \bar{e}_i = \bar{\mathfrak{d}}(\sum_i x_i \bar{x}_i),$$

(supplement)

We should have set up the following assumption (8a-c) and lead (assumption 8) from there:

(8a) $\mathfrak{d}^2 = 0$, $\bar{\mathfrak{d}}^2 = 0$.

(8b) $\mathfrak{d}, \bar{\mathfrak{d}}$ respect the tensor product decomposition by variables:

$$\mathrm{WC}_{n+1} \cong \mathrm{WC}_1^{\otimes n+1}$$

From (1)-(7) and (8a,8b) we deduce that $\mathfrak{d}\bar{e} = \bar{x} + \text{constant}$.

(8c) $\mathfrak{d}\bar{e} = \bar{x}$.

3.2. **choice of the moment map.** In this subsection we discuss what to take as a moment map. It may seem reasonable to adopt, as before

$$\text{(I)} \qquad \mu_{(\mathrm{I})} = \sum_i x_i \bar{x}_i - R$$

where $R \in \Bbbk_1$ is a constant. This corresponds to the following grading of WC.

	x_i	$\bar{x}_i$	e_i	$\bar{e}_i$
$\deg_{(I)}$	1	-1	0	0

Marsden-Weinstein quotient by $\mu_{(I)}$ equals to the quotient ring of the ring of elements of $\deg_{(I)}$-degree 0 by the relation $\mu_{(I)} = 0$. But the relation is not very good for us. We need our ring A to admit the actions of our odd derivations $\mathfrak{d}, \bar{\mathfrak{d}}$. If we admit $\mu_{(I)} = 0$, that means, $\sum_i x_i \bar{x}_i = R$ in A, then by differentiating the equation by $\mathfrak{d}$ or $\bar{\mathfrak{d}}$, we have

$$\sum_i x_i \bar{e}_i = 0, \qquad \sum_i \bar{x}_i e_i = 0.$$

By considering the adjoint by the above elements, we see that $\mathfrak{d}x = 0$, $\bar{\mathfrak{d}}x = 0$ for any element $x \in A$. It is not very interesting.

Instead of (I), it seems, it would be appropriate to take

$$\text{(II)} \qquad \mu_{(\mathrm{II})} = k\sum_i x_i \bar{x}_i + \sum_i e_i \bar{e}_i - \tilde{R}$$

where $\tilde{R}$ is a constant. The adjoint $\operatorname{ad}(k\sum_i x_i \bar{x}_i + \sum_i e_i \bar{e}_i - \tilde{R})$ corresponds, up to a constant multiple, the following grading of WC.

x_i	$\bar{x}_i$	e_i	$\bar{e}_i$
1	-1	1	-1

It is also important to point out that $\mu_{(\mathrm{II})}$ is $\mathfrak{d}$-closed and $\bar{\mathfrak{d}}$-closed. We do not have to struggle with a difficulty seen in $\mu_{(I)}$. Furthermore, we have $[\mathfrak{d}, \bar{\mathfrak{d}}] = \frac{1}{h}\operatorname{ad}(\mu_{(\mathrm{II})})$, so as a consequence, $\mathfrak{d}$ and $\bar{\mathfrak{d}}$ commute on the Marsden-Weinstein quotient by $\mu_{(II)}$. It is very convenient.

Let us now determine the value of our constant $\tilde{R}$. To adjust to the prototype, we may put $\tilde{R} = kR$. Furthermore, by scaling the generators, we see that only constant that affect our theory is a ratio h/R. So by adjusting h if necessary, we may put $R = 1$.

As a conclusion:

Conclusion (affine case)

As the moment map we adopt $k\sum_i x_i\bar{x}_i + \sum_i e_i\bar{e}_i - k$.

Note: We will later use homogeneous coordinates so that we employ a homoegeneous version of our moment map:

$$k\sum_i X_i\bar{X}_i + \sum_i E_i\bar{E}_i - kC.$$

4. SOME BASIC IDEAS

4.1. stereo action and stereo modules.

Definition 4.1.1. Let us consider two commutative rings:

(1) The commutative polynomial ring $\mathrm{Poly} \stackrel{\mathrm{def}}{=} \Bbbk_1[X_0, X_1, \dots, X_n]$ of "unmarked" variables
(2) The other is the commutative polynomial ring $\overline{\mathrm{Poly}} \stackrel{\mathrm{def}}{=} \Bbbk_1[\bar{X}_0, \bar{X}_1, \dots, \bar{X}_n]$ of "bar-ed " variables.

Let us consider an action of $\mathrm{Poly} \otimes \overline{\mathrm{Poly}}$ on WC_{n+1} given by

$$(f_1 \otimes f_2).x = f_1 x f_2 \qquad (f_1 \in \mathrm{Poly}, f_2 \in \overline{\mathrm{Poly}},)x \in \mathrm{WC}_{n+1}$$

and call it the "stereo action".

Via the stereo action our algebra WC_{n+1} may be regraded as a module over a polynomial ring $\Bbbk_1[X_0, X_1, \dots, X_n, \bar{X}_0, \bar{X}_1, \dots \bar{X}_n]$ of $2n+2$-variables. It is a bi-graded module in the usual sense in the commutative algebras.

We may therefore associate WC to a sheaf on $\mathbb{A}^{n+1} \times \bar{\mathbb{A}}^{n+1}$ and also to (by taking a quotient by $\mathbb{G}_m \times \mathbb{G}_m$) a sheaf $\mathcal{WC}$ on $\mathbb{P}^n \times \mathbb{P}^n$.

We need to be careful to make sure that sheaf $\mathcal{WC}$ is merely a sheaf of modules, not of algebras, on $\mathbb{P}^n \times \mathbb{P}^n$. When $\mathrm{char}(\Bbbk) > 0$, however, we may define and understand the multiplicative structure of $\mathcal{WC}$.

The inclusion map $\Bbbk_1[X_0^p, \dots, X_n^p, \bar{X}_0^p, \dots, \bar{X}_n^p] \subset \Bbbk_1[X_0, \dots, X_n, \bar{X}_0, \dots, \bar{X}_n]$ gives a homeomorphism of the corresponding associated projective schemes. We therefore regard the sheaf corresponding to $\Bbbk_1[X_0^p, \dots, X_n^p, \bar{X}_0^p, \dots, \bar{X}_n^p]$ as a subsheaf of $\mathcal{O}_{\mathbb{P}^n}$ and denote it as $\mathcal{O}^{(p)}$.

As the polynomial algebra $\Bbbk_1[X_0^p, \dots, X_n^p, \bar{X}_0^p, \dots, \bar{X}_n^p]$ is contained in the center of WC, we can certainly say that $\mathcal{WC}$ has an algebra structure over $\mathcal{O}^{(p)}$.

4.2. **Change of the order of subs and quotients.** Let us explain a bit about the strategy we take. In order to obtain the final target on $\mathbb{P}^n \times \mathbb{P}^n$ from the homogeneous Weyl-Clifford algebra WC, it is conceivable to take three steps as schematically described below.

$$\mathrm{WC} \overset{1}{\rightsquigarrow} (\mathrm{WC})_{(0)} \overset{2}{\rightsquigarrow} A = (\mathrm{WC})_{(0)}/(\mu) \overset{3}{\rightsquigarrow} \mathcal{A}$$

We will explain this in detail. Please keep in mind, though, that whether to observe "function space" or to observe geometric objects like "spec" reverses the roles of subs and quotients.

In terms of the "function-space-level" it is as follows:

Step 1. Take the invariant part by the anti-diagonal action by $\mathbb{G}_m$.

By "anti-diagonal action" we mean the action given by $\mathbb{A}^{n+1} \times \mathbb{A}^{n+1} \ni (v, w) \mapsto (cv, c^{-1}w)$ $(c \in \mathbb{G}_m)$. We may well write it as

$$X_i \mapsto cX_i, \quad \bar{X}_i \mapsto c^{-1}\bar{X}_i, \quad E_i \mapsto cE_i, \quad \bar{E}_i \mapsto c^{-1}\bar{E}_i, \quad C \mapsto C$$

When we consider commutative (and non-super) version, the ring of anti-diagonal action (ring generated by $\{X_i\bar{X}_j\}$) gives the Segré embedding of $\mathbb{P}^n \times \mathbb{P}^n$, which might be saying to us that we are in a right path.

Step 2. Take quotient by the moment element μ and obtain an algebra A. We have already explained that step1+ step2 is the Marsden Weinstein quotient

Step 3. In order to recover what we have done with the homogenization with C, we consider the sheaf of algebras on $P^n \times \mathbb{P}^n$ corresponding to A.

The same thing in the corresponding geometric object is as follows: Step 1. Take quotient by the diagonal action of $\mathbb{G}_m$.

Step 2. Cut by moment element $\mu = 0$.

Step 3. We consider an sheaf $\mathcal{A}$ of algebras on $\mathbb{P}^n \times \mathbb{P}^n$ corresponding A and descend to the cone to $\mathbb{P}^n \times \mathbb{P}^n$ (take a quotient by $\mathbb{G}_m$)

In any case, what is available is sheaf of algebras $\mathcal{A}$ on $\mathbb{P}^n \times \mathbb{P}^n$. Putting step 1 before step 2 is an advantage of the use of the Marsden-Weinstein quotient, μ generates a relatively small ideal in $\mathrm{WC})_{(0)}$. (in the present case, μ belongs to the center of $(\mathrm{WC})_{(0)}$.

In order to obtain $\mathcal{A}$, it is also possible to change the order of the above procedures and do as follows.

Starting with the Weyl-Clifford algebra WC,

step $1' + 3'$: We take quotient by $\mathbb{G}_m \times \mathbb{G}_m$.

That means, we regard WC as the stereo module and consider the sheaf of algebras $\mathcal{WC}$ on $\mathbb{P}^n \times \mathbb{P}^n$.

step $2'$: We cut the sheaf $\mathcal{WC}$ by $\mu = 0$.

This strategy is more attractive. Step $1'$ is done first, so we can safely perform step $2'$. Therefore, we will follow this strategy.

5. SOME DEFINITIONS

5.1. **open sets** $U, \bar{V}$**.** In later sections, we make some arguments using local coordinates of $\mathbb{P}^n$. To somewhat ease the later notation, we use the following conventions.

As an affine subset of $\mathbb{P}^n$, we select a piece $U_i = \{X_i \neq 0\}$ and call it U. We denote by i_U the selected index i.

$$U = \{X_{i_U} \neq 0\}.$$

Similarly, we define an affine piece $\bar{V}$ of $\bar{\mathbb{P}}^n$ (which is just a copy of $\mathbb{P}^n$).

$$\bar{V} = \{\bar{X}_{i_{\bar{V}}} \neq 0\}.$$

We call the homogeneous coordinates $X_{i_U}, \bar{X}_{i_{\bar{V}}}$ simply $X_U, \bar{X}_{\bar{V}}$ (respectively). The following two sheaves are important for our later arguments.

$$\Omega^{\bullet\bullet}[\partial \log(\bar{X}_{\bar{V}})]$$

$$\Omega^{\bullet\bullet}[d \log(X_U \bar{X}_{\bar{V}})]$$

5.2. **Structure of the sheaf $\mathcal{WC}$.**

5.2.1. $\tilde{\Omega} = \pi_*(\Omega)^{\mathbb{G}_m}$. In this paper $\tilde{\Omega}$, or, what is the same but somewhat more precisely, $\pi_*(\Omega)^{\mathbb{G}_m}$ frequently comes out. It is a coherent sheaf on $\mathbb{P}^n$ with an algebra structure. The symbols are also almost reasonable as described below, but various things have been omitted little by little, so it has become obscure. I will write it clearly here.

Definition 5.2.1. (1) We denote the scheme $\mathbb{A}^{n+1} \setminus \{0\}$ as $\mathbb{A}_o^{n+1}$.
(2) We denote the natural projection $\mathbb{A}_o^{n+1} \to \mathbb{P}^n$ by π.
(3) We denote the de Rham complex on $\mathbb{A}_o^{n+1}$ by $\Omega_{\mathbb{A}_o^{n+1}}$.
(4) By shorthand, we denote a sheaf $\pi_*\Omega_{A_o^{n+1}}$ on $\mathbb{P}^n$ as $\pi_*\Omega$.
(5) As $\pi_*\Omega$ has a natural $\mathbb{G}_m$ action, we denote the sheaf of invariant sections by $(\pi_*\Omega)^{\mathbb{G}_m}$.

5.2.2. *Expression of $(\pi_*\Omega)^{\mathbb{G}_m}$ by coordinates.* Let us take homogeneous coordinates $X_0, X_1, \dots, X_n$ of $\mathbb{P}^n$. We denote by π a projection

$$\pi : \mathbb{A}_o^{n+1} \ni (X_0, X_1 \dots, X_n) \mapsto [X_0, X_1 \dots, X_n]$$

When restricted on an open set $\{X_0 \neq 0\}$ of $\mathbb{A}^{n+1}$, π may be identified with

$$(X_0, X_1, \dots, X_n) \mapsto [1 : X_1/X_0, \dots X_n/X_0].$$

We may thus decompose π in the following way.

$$\begin{array}{ccccc}
\mathbb{G}_m \times \mathbb{A}^n & \hookrightarrow & \mathbb{A}^{n+1} & \stackrel{\pi}{\to} & \mathbb{A}^n \subset \mathbb{P}^n \\
\Cup & & \Cup & & \Cup \\
(c, (x_1, x_2, \dots, x_n)) & \mapsto & (c, cx_1, cx_2, \dots, cx_n) & \mapsto & (x_1, x_2, \dots, x_n)
\end{array}$$

We may take a general index i_U instead of 0. We may then realize:

local expression of $\pi_*\Omega^{\mathbb{G}_m}$

The sheaf $\pi_*\Omega^{\mathbb{G}_m}$ on $\mathbb{P}^n$ is a sheaf of super commutative algebra

$$(\pi_*\Omega)^{\mathbb{G}_m} \cong \Omega_{\mathbb{P}^n}[X_U^{-1} dX_U].$$

An exact sequence satisfied by $\pi_*\Omega^{\mathbb{G}_m}$

$$0 \to \Omega_{\mathbb{P}^n} \stackrel{\iota}{\to} (\pi_*\Omega)^{\mathbb{G}_m} \stackrel{\mathrm{Int}_{\mathrm{Euler}}}{\to} \Omega_{\mathbb{P}^n} \to 0$$

where ι is a injection obtained by the pull back of forms. $\mathrm{Int}_{\mathrm{Euler}}$ denotes the interior product with the Euler operator.

The following is well known. (We take $\mathbb{Z}$ as the coefficient ring since it is the most generic.)

Proposition 5.2.2.

$$H^\bullet(\mathbb{P}^n, \Omega^\bullet) \cong \mathbb{Z}[L]/(L^{n+1})$$

We obtain by an exact sequence: There exists a free generator v_0, v_n of degrees 0 , n (respectively) such that,

$$H^\bullet(\mathbb{P}^n, \widetilde{\Omega^\bullet}) \cong \mathbb{Z}v_0 \oplus \mathbb{Z}v_n.$$

5.2.3. *Structure of* $\mathcal{WC}$. We have two $\mathbb{P}^n$'s at hand, one is "without bar" and the other is "with bar". We may write ∂ as the exterior derivation for the former, and $\bar{\partial}$ as the exterior derivation of the latter.

By studying the ring structure of the Weil-Clifford algebra, we see that we have two copies of $(\pi_*\Omega)^{\mathbb{G}_m}$ as subalgebras in the coherent sheaf $\mathcal{WC}$ over $\mathbb{P}^n \times \mathbb{P}^n$. We need to be take care that though the two subalgebras are certainly closed under multiplication, the commutation relation of the elements of two subalgebras is in general very difficult.

In any case, we can say that $\mathcal{WC}$ carries a stereo module structure over $\pi_*\Omega^{\mathbb{G}_m} \boxtimes \pi_*\bar{\Omega}^{\mathbb{G}_m}$, by defining the multiplication rule as " $\pi_*\Omega^{\mathbb{G}_m}$ from the left and $\pi_*\bar{\Omega}^{\mathbb{G}_m}$ from the right."

Theorem 5.2.3. $\mathcal{WC}$ is locally free as a stereo module of $(\pi_*\Omega)^{\mathbb{G}_m} \boxtimes (\pi_*\Omega)^{\mathbb{G}_m}$):

$$\mathcal{WC} \cong \bigoplus_{l=0}^{\infty} \big((\pi_*\Omega)^{\mathbb{G}_m} \boxtimes (\pi_*\Omega)^{\mathbb{G}_m}\big)(-l,-l\big)$$

5.3. **sparse differential forms.** In this section we define sparse forms, and sheaves Ω_{sparse}, $\widetilde{\Omega}_{\text{sparse}}$ they generate. They are defined to make a shortcut to the theory of Deligne, Illusie and Cartier, by using full use of global projective coordinates.

Let us start by quoting the following Theorem due to Deligne and Illusie.

Theorem 5.3.1. [4] Let $\Bbbk$ be a field of characteristic $p > 0$. Let us assume that a smooth X Spec($\Bbbk$)-scheme is liftable to a Witt ring $W_2(\Bbbk)$. We assume $p > \dim(X)$. (This is our extra assumption, just to make the argument a little bit easier.) Then the lifting determines an isomorphism

$$\varphi_{\bar{X}} : \bigoplus \Omega^i_{X^{(p)}/S}[i] \cong F_*\Omega^\bullet_{X/S}$$

such that $\mathcal{H}^i(\varphi_{\tilde{X}}) = C^{-1}$.

We are going to use the theory in a limited case where $X = \mathbb{P}^n$ or $X = \mathbb{P}^n \times \mathbb{P}^n$. The situation becomes simple since we have global coordinates. Let us take a homogeneous coordinate system $X_0, X_1, \dots, X_n$ of $\mathbb{P}^n$, as we have already done before.

For an affine open subset $U_j = \{X_j \neq 0\}$ of $\mathbb{P}^n$, we take $x_i^{(j)} = X_i/X_j$ as local variables.

We denote by Ω_{sparse} as a subsheaf of Ω generated by $\mathcal{O}^p = \{f^p; f \in \mathcal{O}\}$ and

$$\{(x_i^{(j)})^{p-1}x_i^{(j)}; i = 0, \dots n\}.$$

on U_j. We see easily that this does not depend on the choice of j and that they indeed glue together to form a sheaf.

We also define, as a subsheaf of $\widetilde{\Omega} = \Omega[d\log(X_j)]$, a sheaf $\widetilde{\Omega}_{\text{sparse}}[d\log X_j]$ It is also independent of the choice of j and glue together.

We also note that

$$d\log X_j - d\log X_l = d\log(X_j/X_k) \in \Omega_{\text{sparse}}$$

holds.

The following inverse Cartier operator plays an important role in the theory of Deligne-Illusie-Cartier theory. It is also important in our story.

Proposition 5.3.2. [5, Th 2.1.9] For a scheme X over a scheme S over a field of characteristic p, We put $X^{(p)} = X \times_{S,\text{str},\text{Frob}} S$. Then the inverse Cartier operator

$$C^{-1}_{X/S} : \Omega_{X^{(p)}/S} \cong \mathcal{H}(\Omega_{X/S}),$$

defined by for $f \in X^p$, $f \mapsto f^p$, $df \mapsto f^{p-1}df$, is an isomorphism.

When $X = \mathbb{P}^n$, we may, by using the projective coordinates, lift the Cartier operator and obtain

$$\hat{C}^{-1}_{X/S} : \Omega_{X^{(p)}/S} \to \Omega_{X/S}.$$

The image is the global object Ω_{sparse}.

As a result, we have:

Proposition 5.3.3.

$$\Omega_{\mathbb{P}^n,\text{sparse}} \cong \mathcal{H}(\Omega_{\mathbb{P}^n})$$

$$\widetilde{\Omega}_{\mathbb{P}^n,\text{sparse}} \cong \mathcal{H}(\widetilde{\Omega}_{\mathbb{P}^n})$$

In other words, the sheaf $(\Omega_{\mathbb{P}^n,\text{sparse}}, 0)$ is quasi-isomorphic to $(\Omega_{\mathbb{P}^n}, d)$. The sheaf $(\widetilde{\Omega}_{\mathbb{P}^n,\text{sparse}}, 0)$ is quasi-isomorphic to $(\widetilde{\Omega}_{\mathbb{P}^n}, d)$.

This proposition itself is not a difficult one. It can be proved by arguing locally and by treating polynomials.

5.4. **degree** fdeg **of elements of WC as differential forms.** Let us denote by fdeg the "degree as a form". Namely:

variable	X_i	$\bar{X}_i$	E_i	$\bar{E}_i$	k	C
fdeg	$(0,0)$	(0,0)	(1,0)	$(0,1)$	$(1,1)$	$(0,0)$

As we will see later, WC admits an action ("stereo action") of an algebra sheaf of differential forms, and fdeg is compatible with it.

5.5. **definition of our sheaves of algebras $\mathcal{A}$, $\mathfrak{A}$, $\mathfrak{W}C$ on $\mathbb{P}^n \times \mathbb{P}^n$.**

Definition 5.5.1. Let us put $\mu_{(k,0)} \overset{\text{def}}{=} (k\sum_i X_i\bar{X}_i + \sum_i E_i\bar{E}_i)$ and define

$$\mathcal{A} \overset{\text{def}}{=} \mathcal{WC}/(\mu_{(k,0)} - kC)$$

We note that an equation

$$k^p \sum_i X_i^p \bar{X}_i^p = \mu_{(k,0)}^p - (khC)^{p-1}\mu_{(k,0)} = k^p(1-h^{p-1})C^p$$

holds on $\mathcal{A}$. That means, we have necessarily an equation $k^p((\sum_i X_i^p\bar{X}_i^p) - (1-h^{p-1})C^p) = 0$.

So let us take algebras who has the property $(\sum_i X_i^p\bar{X}_i^p) - (1 - h^{p-1})C^p) = 0$. Namely,

Definition 5.5.2. We define

$$\begin{aligned}\mathfrak{A} &\overset{\text{def}}{=} \mathcal{WC}/(\mu_{(k,0)} - kC, \sum_i X_i^p \bar{X}_i^p - (1-h^{p-1})C^p) \\ &(\cong \mathcal{A}/(\sum_i X_i^p \bar{X}_i^p - (1-h^{p-1})C^p)), \\ \mathfrak{WC} &\overset{\text{def}}{=} \mathcal{WC}/(\sum_i X_i^p \bar{X}_i^p - (1-h^{p-1})C^p).\end{aligned}$$

$$\mathfrak{WC} = \bigoplus_{l=0}^{p-1} \widetilde{\Omega[k]} \boxtimes \widetilde{\bar{\Omega}[k]}(-l,-l)$$

6. MAIN RESULT

6.1. **A supplement on spectral sequences.** In this paper we are using spectral sequences to deal with something of a "derived category of a derived category. Let us briefly state a supplementary result we use on spectral sequences.

Proposition 6.1.1. Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories. We assume $\mathcal{C}_1$ has enough injectives. Let $M^{\bullet\bullet}$ a double complex of objects in $\mathcal{C}_1$. We assume M to be bounded below in the sense that there exists an integer k such that $M^{ij} = 0$ whenever $i < k$ or $j < k$. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a left-exact additive functor. Then there exists an spectral sequence

$$E_2 = R^i_{d_1} F(H^j_{d_2}(M)) \implies E_\infty = R^{i+j} F(\mathrm{Tot}(M^{\bullet\bullet})).$$

Proof. Let us regard the double complex $(M^{\bullet\bullet})$ as a single d_2-chain complex $(M, d_1)^\bullet$ of d_1-graded modules, that means, a d_2-chain complex of $C_{\mathrm{bdd}}(\mathcal{C}, d_1)$. We take its Cartan-Eilenberg resolution $(M^\bullet, d_1)^\bullet \to (I^\bullet, d_1)^{\bullet\bullet}$ with an extra care: Looking at short exact short exact sequences

$$0 \to \mathrm{Image}\, d_2 \to \mathrm{Ker}\, d_2 \to H_{d_2}(M) \to 0$$

and

$$0 \to \mathrm{Ker}\, d_2 \to M \to \mathrm{Image}(d_2) \to 0,$$

we make use of cones of injective resolutions of Image d_2 and $H_{d_2}(M)$ to create injective resolutions of Ker d_2 and M. We then obtain a Cartan-Eilenberg injective resolution $I^{\bullet\bullet\bullet}$ such that $H_{d_2}(I^{\bullet\bullet\bullet})$ is a injective resolution of $H_{d_2}(M)$. Using the resolution I, we have

$$\begin{aligned} & R^i F(H_{d_2}(M)) \\ \cong & H^i(F(\mathrm{Tot}_{1,3}(H_{d_2}(I^{\bullet\bullet\bullet})))) && (H_{d_2}(I) \text{ is a resolution of } H_{d_2}(M)) \\ = & H^i(\mathrm{Tot}_{1,3}(H_{d_2}(F(I^{\bullet\bullet\bullet})))) && (F \text{ is left exact and } I \text{ is injective}) \\ = & H^i H_{d_2}(\mathrm{Tot}_{1,3}(F(I^{\bullet\bullet\bullet}))) && (\mathrm{Tot}_{1,3} \text{ commutes with } H_{d_2}.) \end{aligned}$$

On the other hand we have

$$\begin{aligned} & R^i F(\mathrm{Tot}(M^{\bullet\bullet})) \\ \cong & H^i(F(\mathrm{Tot}_{123}(I^{\bullet\bullet\bullet}))) && (\mathrm{Tot}(I) \text{ is an injective resolution of } \mathrm{Tot}(M)) \\ \cong & H^i(\mathrm{Tot}_{123}(F(I^{\bullet\bullet\bullet}))) && (F \text{ commutes with Tot}) \\ \cong & H^i(\mathrm{Tot}\,\mathrm{Tot}_{1,3}(F(I^{\bullet\bullet\bullet}))). \end{aligned}$$

So by using an ordinary theory of spectral sequence on a double complex $\mathrm{Tot}_{1,3} F(I^{\bullet\bullet\bullet})$, we obtain the desired spectral sequence. □

6.2. **Statement of the main theorem.** Throughout the paper, what we denote as $\mathcal{F}_1 \boxtimes_? \mathcal{F}_2$ should actually be denoted as $\pi_1^*\mathcal{F}_1 \otimes_? \pi_2^*\mathcal{F}_2$. In most of the cases the coefficient ring ? is equal to $\mathcal{O}^{(p)}$, which we omit.

Recall that we have constructed a sheaf $\mathcal{WC}$ on $\mathbb{P}^n \times \mathbb{P}^n$ as a sheaf associated to the homogeneous Weyl-Clifford algebra WC. We also have sheaves $\mathfrak{A}, \mathfrak{WC}, \mathcal{A}$ as its quotients. They have derivation $\bar{\mathfrak{d}}$ defined as

$$\bar{\mathfrak{d}} = -\frac{1}{hC}\,\mathrm{ad}(\bar{\varepsilon})$$

where $\varepsilon = \sum X_i \bar{E}_i$ (subsubsection2.3.4). (We have explained some reason for this choice in section3.1.)

Theorem 6.2.1. In the following we assume the charcteristic of the base field $\Bbbk$ is $p > 0$. We also assume p is greater than n, the size of our Weyl-Clifford algebras.

(1) Sheaves $\mathcal{WC}, \mathcal{A}$ have structure of $\mathcal{O}^{(p)}$-algebras. We may define $\mathfrak{A}, \mathfrak{WC}$ as their quotient algebras.
(2) $\mathcal{WC}, \mathcal{A}, \mathfrak{A}, \mathfrak{WC}$ are double complexes with odd differentials $\mathfrak{d}, \bar{\mathfrak{d}}$.
(3) As stereo modules, $\mathcal{WC} \cong \bigoplus_{l\geq 0} \widetilde{\Omega^\bullet[k]} \boxtimes_{\mathcal{O}^{(p)}[k]} \widetilde{\bar{\Omega}^\bullet[k]} \otimes (\mathcal{O}(-l,-l))$
(4) As $\bar{\mathfrak{d}}$-complex, $(\widetilde{\Omega^\bullet[k]}, \bar{\mathfrak{d}}) \cong (\widetilde{\Omega^\bullet[k]}, -kI_0)$, $(\widetilde{\bar{\Omega}^\bullet[k]}, \bar{\mathfrak{d}}) \cong (\widetilde{\bar{\Omega}^\bullet[k]}, \bar{\partial})$ where I_0 denotes the interior product with the Euler vector field $\sum_i X_i d/dX_i$.
(5) The sheaf of $\bar{\mathfrak{d}}$-hyper cohomology groups are given as follows
 (i) For $\mathcal{WC}$:

$$\begin{aligned}(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathcal{WC}), \mathfrak{d}) &\cong (\Omega^\bullet, \partial) \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}[C^p_{U\bar{V}}], 0)\\ &\overset{\mathfrak{d}-\text{q.i}}{\sim} (\bigoplus_{l\geq 0}(\Omega^\bullet_{\text{sparse}} \boxtimes_{\mathcal{O}^{(p)}} (\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}[C^p_{U\bar{V}}], 0)\end{aligned}$$

We have in particular,

$$\mathcal{H}_{\mathfrak{d}}(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathcal{WC})) \cong \Omega^\bullet_{\text{sparse}} \boxtimes_{\mathcal{O}^{(p)}} (\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}[C^p_{U\bar{V}}]$$

$$R^\bullet\Gamma(\mathcal{H}^j_{\bar{\mathfrak{d}}}(\mathcal{WC}), \mathfrak{d}) \cong \bigoplus_{l=0}^{\infty} H^\bullet(\mathbb{P}^n \times \mathbb{P}^n, \Omega^\bullet_{\text{sparse}} \otimes_{\mathcal{O}^{(p)}} \widetilde{\bar{\Omega}}^\bullet_{\text{sparse}}(-lp, -lp))$$

 (ii) For $\mathcal{A}$:

$$\begin{aligned}(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathcal{A}), \mathfrak{d}) \cong&(\Omega^\bullet, \partial) \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}, 0)\\ &\oplus (\widetilde{\Omega}_{(k=0)}, \partial) \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)})[C^p_{U\bar{V}}]C^p_{U\bar{V}}, 0)\\ &\overset{\mathfrak{d}-\text{q.i}}{\sim}(\Omega^\bullet_{\text{sparse}}, 0) \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}, 0)\\ &\oplus (\widetilde{\Omega_{\text{sparse}(k=0)}}, 0) \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)})[C^p_{U\bar{V}}]C^p_{U\bar{V}}, 0)\end{aligned}$$

In particular,

$$\begin{aligned}\mathcal{H}_{\mathfrak{d}}(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathcal{A})) \cong&\Omega^\bullet_{\text{sparse}} \boxtimes_{\mathcal{O}^{(p)}} (\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}\\ &\oplus \widetilde{\Omega_{\text{sparse}(k=0)}} \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)})[C^p_{U\bar{V}}]C^p_{U\bar{V}}\end{aligned}$$

$$\begin{aligned}R^\bullet\Gamma(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathcal{A})) \cong&H^\bullet(\Omega^\bullet_{\text{sparse}} \boxtimes_{\mathcal{O}^{(p)}} (\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)})\\ &\oplus \bigoplus_{l=1}^{\infty} H^\bullet(\widetilde{\Omega_{\text{sparse}(k=0)}} \boxtimes_{\mathcal{O}^{(p)}} ((\widetilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)})(-lp, -lp))\end{aligned}$$

(iii) For $\mathfrak{WC}$, $\mathfrak{A}$:

$$\mathcal{H}_{\mathfrak{d}}(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathfrak{WC})) \cong \mathcal{H}_{\mathfrak{d}}(\mathcal{H}_{\bar{\mathfrak{d}}}(\mathfrak{A})) \cong \Omega^\bullet_{\text{sparse}} \boxtimes_{\mathcal{O}^{(p)}} (\tilde{\bar{\Omega}}^\bullet_{\text{sparse}})_{(k=0)}$$

$$R^\bullet\Gamma(\mathcal{H}^j_{\bar{\mathfrak{d}}}(\mathfrak{WC}), \mathfrak{d}) \cong R^\bullet\Gamma(\mathcal{H}^j_{\bar{\mathfrak{d}}}(\mathfrak{A}), \mathfrak{d}) \cong H^\bullet(\mathbb{P}^n \times \mathbb{P}^n, \Omega^\bullet_{\text{sparse}} \otimes_{\mathcal{O}^{(p)}} \tilde{\bar{\Omega}}^\bullet_{\text{sparse}})$$

(6) Let us put $\mathbf{d} = \mathfrak{d} + \bar{\mathfrak{d}}$. Then:

$$(\mathcal{WC}, \mathbf{d}) \overset{q.i}{\sim} \bigoplus_{l \geq 0} ((\Omega^\bullet_{\text{sparse}} \otimes \bar{\Omega}^\bullet_{\text{sparse}}[d\log(X_U \bar{X}_{\bar{V}})(-lp, -lp)), 0)$$

We have therefore:

$$\mathcal{H}^i(\mathcal{WC}, \mathbf{d}) \cong \bigoplus_{l \geq 0} (\Omega^\bullet_{\text{sparse}} \otimes \bar{\Omega}^\bullet_{\text{sparse}}[d\log(X_U \bar{X}_{\bar{V}})(-lp, -lp))$$

$$R^i\Gamma(\mathcal{WC}, \mathbf{d}) \cong \bigoplus_{l \geq 0} R^i\Gamma(\mathbb{P}^n \times \mathbb{P}^n, \Omega^\bullet_{\text{sparse}} \boxtimes \bar{\Omega}^\bullet_{\text{sparse}}[d\log(X_U \bar{X}_{\bar{V}})](-lp, -lp))$$

Corollary 6.2.2. Let us consider the following spectral sequence obtained in Proposition 6.1.1: spectral sequence

$$\text{(A)} \qquad \begin{aligned} E_2 &= R_{\mathfrak{d}}\Gamma(\mathbb{P}^n \times \mathbb{P}^n; \mathcal{H}_{\bar{\mathfrak{d}}}(\mathfrak{WC})) \\ &\Longrightarrow E_\infty = R^\bullet_{\mathfrak{d}+\bar{\mathfrak{d}}}\Gamma(\mathbb{P}^n \times \mathbb{P}^n, (\mathfrak{WC})). \end{aligned}$$

Then:

(1) The spectral sequence (A) is isomorphic to the following.

$$\begin{aligned} E_2 &= H^\bullet(\mathbb{P}^n, \Omega_{\text{sparse}}) \otimes H^\bullet(\mathbb{P}^n, \Omega_{\text{sparse}}[d\log \bar{X}_V]) \\ &\Longrightarrow E_\infty = R^i\Gamma(\mathbb{P}^n \times \mathbb{P}^n, \Omega^\bullet_{\text{sparse}} \boxtimes \Omega^\bullet_{\text{sparse}}[d\log(X_U \bar{X}_V)]) \end{aligned}$$

(2) The cohomology groups are computed so

$$E_2 \cong \Bbbk[L_2, \bar{L}_n]/(L_2^n, \bar{L}_n^2)$$

By computing the dimensions, we see that $\dim E_2 = \dim E_\infty = 2n$ so that (A) degenerates.

As our original theory naturally has a $\bullet \leftrightarrow \bar{\bullet}$ symmetry, we may expect we have extra bar-structure in the cohomology group.

6.3. **proof of the main theorem.** (1), (2) are direct consequences of the definition. (3), (4) are directly verified by calculation.

Let us proceed to the proof of (5). Since $\mathcal{H}_{\bar{\mathfrak{d}}}$ pass through the derived category (of graded $\mathfrak{d}$-complexes), we consider the story in that place.

(i)

Let us put $c_{U\bar{V}} = X_U^{-1} C \bar{X}_{\bar{V}}$ and see how to deal with it. We have a direct-sum decomposition

$$\text{(DS)} \qquad \mathcal{WC} \cong \bigoplus_{l \geq 0} \widetilde{\Omega[k]} \boxtimes_{\Bbbk_1[k]} \widetilde{\bar{\Omega}[k]} c^l_{U\bar{V}}$$

.

$$\begin{aligned} \bar{\mathfrak{d}} c^l_{U\bar{V}} &= \bar{\mathfrak{d}}(X_U^{-l} C^l \bar{X}_V^{-l}) \\ &= X_U^{-l} C^l \cdot (-l) \bar{X}_V^{-l-1} \bar{E}_V) \\ &= -l X_U^{-l} C^l \bar{X}_V^{-l} X_V^{-1} E_V \\ &= -l c^l_{U\bar{V}} (\bar{\partial} \log(\bar{X}_V)) \end{aligned}$$

so $\bar{\mathfrak{d}}$ preserves the direct sum decomposition (DS).

Let us take a section α of $\widetilde{\Omega[k]} \boxtimes_{\mathbb{k}_1[k]} \widetilde{\bar{\Omega}[k]}$ and see how the $\bar{\mathfrak{d}}$-cocycle condition for $\alpha c^l_{U\bar{V}}$ look like.

We present α as $\alpha = \alpha_1 + (\bar{\partial}\log(\bar{X}_V))\alpha_2$ with $\alpha_1, \alpha_2 \in \widetilde{\Omega[k]} \boxtimes_{\mathbb{k}_1[k]} \bar{\Omega}[k]$. Then we have

$$\begin{aligned}
&\bar{\mathfrak{d}}(\alpha c^l_{U\bar{V}}) = 0\\
\Leftrightarrow &(\bar{\partial}\log(\bar{X}_V))\bar{\mathfrak{d}}\alpha_2 - l(\bar{\partial}\log(\bar{X}_V))\alpha_1 = 0\\
\Leftrightarrow &\begin{cases}\bar{\mathfrak{d}}\alpha_1 = 0\\ \bar{\mathfrak{d}}\alpha_2 = -l\alpha_1.\end{cases}
\end{aligned}$$

In particular, if $l \neq 0 \pmod p$, then

$$\begin{aligned}
&\bar{\mathfrak{d}}(\alpha c^l_{U\bar{V}}) = 0\\
\Longrightarrow\ &\alpha_1 = \frac{-1}{l}\bar{\mathfrak{d}}\alpha_2\\
\Longrightarrow\ &\alpha c^l_{U\bar{V}} = (\frac{1}{l})(\bar{\partial}\log(\bar{X}_0))\alpha_2)c^l_{U\bar{V}}\\
&\qquad = \bar{\mathfrak{d}}(\frac{1}{l}\alpha_2 c^l_{U\bar{V}})
\end{aligned}$$

So parts with $l \neq 0 \pmod p$ do not affect the cohomology. $\mathcal{WC}$ is $\bar{\mathfrak{d}}$-quasi isomorphic to $\bigoplus_{l\geq 0} \widetilde{\Omega[k]} \boxtimes_{\mathbb{k}_1[k]} \widetilde{\bar{\Omega}[k]} c^{pl}_{U\bar{V}}$.

$$\mathcal{WC} \cong \bigoplus_{l=0}^{\infty} \widetilde{\Omega[k]} \boxtimes \widetilde{\bar{\Omega}[k]} c^l_{U\bar{V}} \overset{q.i.}{\sim} \bigoplus_{l=0}^{\infty} \widetilde{\Omega[k]} \boxtimes \widetilde{\bar{\Omega}[k]} c^{pl}_{U\bar{V}}$$

Let us use the Deligne-Illusie-Cartier on the right hand side. $(\widetilde{\bar{\Omega}[k]}, \bar{\partial}) \cong (\widetilde{\bar{\Omega}_{\text{sparse}}[k]}, 0)$ (homological isom.) Since sheaves $\widetilde{\bar{\Omega}[k]}$, $\widetilde{\bar{\Omega}_{\text{sparse}}[k]}$ are both flat over $\mathcal{O}^{(p)}[k]$, they are flat resolutions of themselves. The $\otimes$ (hidden in "$\boxtimes$"...) are actually equal to $\overset{\mathbb{L}}{\otimes}$. We may thus deal with them in the framework of our derived category.

$$\mathcal{WC} \overset{q.i}{\sim} \widetilde{\Omega[k]} \boxtimes_{\mathcal{O}^{(p)}[k]} \widetilde{\bar{\Omega}_{\text{sparse}}[k]}[c^p_{U\bar{V}}]$$

where $\widetilde{\bar{\Omega}_{\text{sparse}}[k]}$ is a complex whose derivation $\bar{\partial}$ is equal to 0. By the flatness we may commute tensor products and cohomologies to obtain:

$$\begin{aligned}
\mathcal{H}_{\bar{\mathfrak{d}}}(\mathcal{WC}) &\cong \mathcal{H}_{\bar{\mathfrak{d}}}(\widetilde{\Omega[k]}) \boxtimes_{\mathcal{O}^{(p)}[k]} \widetilde{\bar{\Omega}_{\text{sparse}}[k]}[c^p_{U\bar{V}}]\\
&\cong \Omega \boxtimes_{\mathcal{O}^{(p)}[k]} \widetilde{\bar{\Omega}_{\text{sparse}}[k]}[c^p_{U\bar{V}}]\\
&\cong \Omega \boxtimes_{\mathcal{O}^{(p)}} \widetilde{\bar{\Omega}_{\text{sparse}}}[k][c^p_{U\bar{V}}]\\
&\cong \Omega \boxtimes_{\mathcal{O}^{(p)}} (\widetilde{\bar{\Omega}}^{\bullet}_{\text{sparse}})_{(k=0)}[c^p_{U\bar{V}}]
\end{aligned}$$

(ii) $\mathcal{WC}/(kC-\mu_0) \overset{q.i.}{\sim} \text{Cone}(\mathcal{WC}[1] \overset{kC-\mu_0}{\to} \mathcal{WC})$. Since we have $\mu_0 = \bar{\mathfrak{d}}(\mathfrak{d}(F))$ and $F = \sum_i X_i\bar{X}_i$, the complex $(\mathcal{WC}[1] \overset{kC-\mu_0}{\to} \mathcal{WC})$ is $\bar{\partial}$ -homotopic to the complex $(\mathcal{WC}[1] \overset{kC}{\to} \mathcal{WC})$. (Note that, since we have $\mathfrak{d}^2 = 0$, the homotopy $\mathfrak{d}(F)$ super commutes to $\mathfrak{d}$.) We have therefore, $\mathcal{WC}/(kC-\mu_0) \overset{q.i.}{\sim} \text{Cone}(\mathcal{WC}[1] \overset{kC}{\to} \mathcal{WC}) \overset{q.i.}{\sim} \mathcal{WC}/(kC)$. In other words, we may treat $\mathcal{WC}/(kC)$ instead of $\mathcal{A}$. Locally speaking, we have:

$$
\begin{aligned}
\mathcal{WC}/(kC) &\cong \widetilde{\Omega[k]} \boxtimes \widetilde{\bar{\Omega}[k]} \oplus \bigoplus_{l>0} (\widetilde{\Omega[k]} \boxtimes \widetilde{\bar{\Omega}[k]}/(k)) c^l_{U\bar{V}} \\
&\overset{q.i.}{\sim} \Omega \boxtimes \widetilde{\bar{\Omega}[k]}/(k) \oplus \bigoplus_{l>0} (\tilde{\Omega} \boxtimes \tilde{\bar{\Omega}}_{\text{sparse}}) c^l_{U\bar{V}} \\
&\overset{q.i.}{\sim} \Omega \boxtimes \widetilde{\bar{\Omega}_{\text{sparse}}[k]}/(k) \oplus \bigoplus_{l>0} (\tilde{\Omega} \boxtimes \tilde{\bar{\Omega}}_{\text{sparse}}) c^l_{U\bar{V}} \\
&\overset{q.i.}{\sim} \Omega \boxtimes \widetilde{\bar{\Omega}_{\text{sparse}}[k]}/(k) \oplus \bigoplus_{l>0} (\tilde{\Omega} \boxtimes \tilde{\bar{\Omega}}_{\text{sparse}}) c^{pl}_{U\bar{V}}
\end{aligned}
$$

(iii)Every sheaf which appears when dealing with $\mathcal{A}$, WC was flat over $\mathbb{P}^n \times \mathbb{P}^n$. So we may tensor (something) and obtain

$$
\begin{aligned}
\mathfrak{W}C &\overset{q.i.}{\sim} \bigoplus_{l\geq 0} (\Omega \boxtimes_{\mathcal{O}^{(p)}} (\tilde{\bar{\Omega}}^{\bullet}_{\text{sparse}})_{(k=0)}[c^p_{U\bar{V}}]) \otimes_{\mathcal{O}^{(p)}} (\mathcal{O}^{(p)}/(C^p - (1-h^{p-1})F^p)) \\
&\cong \Omega \boxtimes_{\mathcal{O}^{(p)}} (\tilde{\bar{\Omega}}^{\bullet}_{\text{sparse}})_{(k=0)}
\end{aligned}
$$

(iv) Likewise, we have

$$\mathfrak{A} \overset{q.i.}{\sim} \Omega \boxtimes \widetilde{\bar{\Omega}_{\text{sparse}}[k]}/(k)$$

(6) is obtained by another application of Deligne-Illusie-Cartier theory.

(7) follows from (6).

(8) likewise for d.

Let us put $b_U = \partial \log(X_U)), \bar{b}_{\bar{V}} = \bar{\partial} \log(\bar{X}_0))$. Note that $d(b_U + \bar{b}_{\bar{V}}) = 0$.

For $\alpha \in \widetilde{\Omega[k]} \boxtimes \widetilde{\bar{\Omega}[k]}$, let us put

$$\alpha = \beta_0 + (b_U + \bar{b}_{\bar{V}})\beta_1.$$

$(\beta_0, \beta_1 \in \Omega \boxtimes \bar{\Omega}[k, b_U])$.

$$
\begin{aligned}
d(\alpha c^l_{U\bar{V}}) =&(d\beta_0 - (b_U + \bar{b}_{\bar{V}})d\beta_1)c^l_{U\bar{V}} - l(b_U + \bar{b}_{\bar{V}})\beta_0 c^l_{U\bar{V}} \\
=&(d\beta_0)c^l_{U\bar{V}} + (b_U + \bar{b}_{\bar{V}})(-d\beta_1 - l\beta_0)c^l_{U\bar{V}}
\end{aligned}
$$

$\mathfrak{d}, \bar{\mathfrak{d}}$ (and their sum d also) preserves the number of $b_U, \bar{b}_{\bar{V}}$. We my thus compare the coefficients of $(b_U + \bar{b}_{\bar{V}})$ and obtain

$$d(\alpha c^l_{U\bar{V}}) = 0 \Leftrightarrow \begin{cases} d\beta_0 = 0 \\ d\beta_1 = -l\beta_0 \end{cases}$$

As a result terms of $c^l_{U\bar{V}}$ which do not satisfy $p|l$ do not affect d-cohomology.

If $p|l$, then $c^l_{U\bar{V}}$ is d-closed. For $\alpha(k), \beta(k) \in \Omega \boxtimes \bar{\Omega}$, we have

$$
\begin{aligned}
&d(b_U\alpha(k) + \beta(k)) = 0 \\
\Leftrightarrow&(-b_U d\alpha(k) + k\alpha(k) + d\beta(k) = 0 \\
\Leftrightarrow&\begin{cases} d\alpha(k) = 0 \\ k\alpha(k) + d\beta(k) = 0. \end{cases}
\end{aligned}
$$

Non-constant terms of k in β is unaffected. When we divide d-closeds by a d-exact, then what we get is equal to $\mathcal{H}(\Omega\boxtimes\bar{\Omega})$. With the Deligne-Illusie-Cartier theory, we obtain:

$(\bigoplus_{l\geq 0}\Omega_{\mathrm{sparse}}\boxtimes\bar{\Omega}_{\mathrm{sparse}}(-pl,-pl),d)$ is quasi isomorphic to $(\mathcal{WC},d)$.

6.4. **The spectral sequence.**

$$\bigoplus_l H^l(\mathbb{P}^n\times\mathbb{P}^n,\Omega\otimes\tilde{\tilde{\Omega}})=\bigoplus_{l,m} H^l(\mathbb{P}^n,\Omega)\otimes H^m(\mathbb{P}^n,\tilde{\tilde{\Omega}})\cong\Bbbk[L]\otimes\Bbbk[v_n]$$

(L is an element of degree 2 with $L^{n+1}=0$i. v_n is an element of degree $2n$ with $v_n^2=0$.) We have:

$$\sum_l \dim(H^l(\mathbb{P}^n\times\mathbb{P}^n,\Omega\otimes\tilde{\tilde{\Omega}}))=2n+2.$$

On the other hand, let us put $M=\oplus_l R^l\Gamma(\widetilde{\Omega\otimes\bar{\Omega}},d)$.

$$0\to\Omega\otimes\bar{\Omega}\to\widetilde{\Omega\otimes\bar{\Omega}}\to\Omega\otimes\bar{\Omega}\to 0\quad:\text{exact}$$

We have therefore $0\to H^\bullet(\mathbb{P}^n\times\mathbb{P}^n,\Omega\otimes\bar{\Omega})/\operatorname{Image}(L+\bar{L})\to M\to\operatorname{Ker}(L+\bar{L})\to 0:$ exact.

The dimension of $\Bbbk[L]$ is $(n+1)$.

$0\to\operatorname{Ker}(L+\bar{L})\to\Bbbk[L,\bar{L}]\to(L+\bar{L})\Bbbk[L,\bar{L}]\to 0$:exact

and

$0\to(L+\bar{L})\Bbbk[L,\bar{L}]\to\Bbbk[L,\bar{L}]\overset{\text{subtraction}}{\to}\Bbbk[L]\to 0:$ exact

implies

$$\dim\operatorname{Ker}(L+\bar{L})=(n+1)^2-(n+1)=n^2+n.$$

$0\to(L+\bar{L})\Bbbk[L,\bar{L}]\to\Bbbk[L,\bar{L}]\overset{\text{subtraction}}{\to}\Bbbk[L]\to 0:$ exact so that $\dim(M)=2n+2$.

We conclude therefore that our spectral sequence (A) degenerates.

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