No.10: The ring of p-adic Witt vectors revisited

LEMMA 10.1. *Let A be a commutative ring. Then:*

(1) *For any $a, b \in A$, we have*

$$[a] \cdot [b] = [ab]$$

(2) *If $a \in A$ satisfies $a^q = a$ for some positive integer q, then we have*

$$[a]^{\cdot q} = [a].$$

(3) *Let q be a positive integer. If $b \in A$ satisfies*

$$\forall n \in \mathbb{Z}_{>0} \exists b_n \in A \text{ such that } b_n^{q^n} = b,$$

then we have

$$\forall n \in \mathbb{Z}_{>0} \exists c_n \in \mathcal{W}_1(A) \text{ such that } c_n^{q^n} = [b].$$

□

Recall that the ring of p-adic Witt vectors is a quotient of the ring of universal Witt vectors. We have therefore a projection $\varpi : \mathcal{W}_1(A) \to \mathcal{W}^{(p)}(A)$. But in the following we intentionally omit to write ϖ.

PROPOSITION 10.2. *Let p be a prime number. Let A be a ring of characteristic. Then:*

(1) *Every element of $\mathcal{W}^{(p)}(A)$ is written uniquely as*

$$\sum_{j=0}^{\infty} V_p^j([x_j]) \qquad (x_j \in A).$$

(2) *For any $x, y \in A$, we have*

$$V_p^n([x]) \cdot V_p^m([y]) = V_p^{n+m}([x^{p^m} y^{p^n}]).$$

(3) *A map*

$$\varphi : \mathcal{W}^{(p)}(A) \ni \sum_{j=0}^{\infty} V_p^n([x_j]) \mapsto x_0 \in A$$

is a ring homomorphism from $(\mathcal{W}^{(p)}, +, \cdot)$ to $(A, +, \times)$.

(4) $\mathrm{Ker}(\varphi) = \mathrm{Image}(V_p)$.

(5) *An element $x \in \mathcal{W}^{(p)}$ is invertible in $\mathcal{W}^{(p)}$ if and only if $\varphi(x)$ is invertible in A.*

□

COROLLARY 10.3. *If k is a field of characteristic $p \neq 0$, then $\mathcal{W}^{(p)}$ is a local ring with the residue field k. If furthermore the field k is* **perfect** *(that means, every element of k has a p-th root in k), then every non-zero element of $\mathcal{W}^{(p)}$ may be writen as*

$$p^k \cdot x \qquad (k \in \mathbb{N}, x \in (\mathcal{W}^{(p)})^{\cdot} \text{ (i.e. } x\text{:invertible)})$$

Since any integral domain can be embedded into a perfect field, we deduce the following

Corollary 10.4. *Let A be an integral domain of characteristic $p \neq 0$. Then $\mathcal{W}^{(p)}(A)$ is an integral domain of characteristic* 0.

Proof. $\mathcal{W}^{(p)}(\iota)$ is always an injection when ι is. $\square$