CONGRUENT ZETA FUNCTIONS. NO.3

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projective space and projective varieties.

DEFINITION 3.1. Let R be a ring. A polynomial $f(X_0, X_1, \dots, X_n) \in R[X_0, X_1, \dots, X_n]$ is said to be **homogenius** of degree d if an equality

$$f(\lambda X_0, \lambda X_1, \dots, \lambda X_n) = \lambda^d f(X_0, X_1, \dots, X_n)$$

holds as a polynomial in $n+2$ variables $X_0, X_1, X_2, \dots, X_n, \lambda$.

For any homogeneous polynomial $F(X_0, X_1, \dots, X_n)$, we may obtain its inhomogenization as follows:

$$f(x_1, x_2, \dots, x_n) = F(1, X_1, \dots, X_n).$$

Conversely, for any inhomogeneous polynomial $f(x_1, \dots, x_n)$ of degree d, we may obtain its homogenization as follows:

$$F(X_1, X_2, \dots, X_n) = f(X_1/X_0, \dots, X_n/X_0)X_0^d.$$

DEFINITION 3.2. Let k be a field.

(1) We put

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times$$

and call it (the set of k-valued points of) the **projective space**. The class of an element $(x_0, x_1, \dots, x_n)$ in $\mathbb{P}^n(k)$ is denoted by $[x_0 : x_1 : \dots : x_n]$.

(2) Let $f_1, f_2, \dots, f_l \in k[X_0, \dots, X_n]$ be homogenious polynomials. Then we set

$$V_h(f_1, \dots, f_l) = \{[x_0 : x_1 : x_2 : \dots x_n]; f_j(x_0, x_1, x_2, \dots, x_n) = 0 \qquad (j = 1, 2, 3, \dots, l)\}.$$

and call it (the set of k-valued point of) the **projective variety** defined by $\{f_1, f_2, \dots, f_l\}$.

(Note that the condition $f_j(x) = 0$ does not depend on the choice of the representative $x \in k^{n+1}$ of $[x] \in \mathbb{P}^n(k)$.)

LEMMA 3.3. *We have the following picture of* $\mathbb{P}^2$.

(1)

$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1.$$

That means, $\mathbb{P}^2$ *is divided into two pieces* $\{Z \neq 0\} = \complement V_h(Z)$ *a nd* $V_h(Z)$.

(2)

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$

That means, $\mathbb{P}^2$ *is covered by three "open sets"* $\{Z \neq 0\}, \{Y \neq 0\}, \{X \neq 0\}$. *Each of them is isomorphic to the plane (that is, the affine space of dimension 2).*