CONGRUENT ZETA FUNCTIONS. NO.8

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plane conic

Definition 8.1. Let k be a field. A **projective transformation** of $\mathbb{P}^n = \mathbb{P}^n(k)$ is a map

$$f : \mathbb{P}^n \to \mathbb{P}^n$$

which is given by a non-degenerate matrix $A \in \mathrm{GL}_{n+1}(k)$ as follows:

$$f([v]) = [A.v] \qquad (v \in k^{n+1})$$

where [v] is the class of $v \in k^{n+1}$ in $\mathbb{P}^n$.

We would like to prove the following proposition.

Proposition 8.2. *Let $F = F(X, Y, Z) \in \mathbb{F}_q[X, Y, Z]$ be a homogenious polynomial of degree 2. We assume F is irreducible over $\overline{\mathbb{F}_q}$. Let us put $C = V_h(F)$. Then:*

1. *There exists at least one $\mathbb{F}_q$-valued point P in $V_h(F)$.*
2. *For any line L passing through P defined over $\mathbb{F}_q$, the intersection $L \cap C$ consists of two $\mathbb{F}_q$-valued points P and Q_L except for a case where L contacts C.*
3. *There exists a projective change of coordinate $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f(V_h(F)) = V_h(XY - Z^2)$.*
4. *The congruent zeta function of C is always equal to the congruent zeta function of $\mathbb{P}^1$.*

Lemma 8.3. *We have the following picture of $\mathbb{P}^2$.*

(1)
$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1.$$
That means, $\mathbb{P}^2$ is divided into two pieces $\{Z \neq 0\} = \complement V_h(Z)$ and $V_h(Z)$.

(2)
$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$
That means, $\mathbb{P}^2$ is covered by three "open sets" $\{Z \neq 0\}, \{Y \neq 0\}, \{X \neq 0\}$. Each of them is isomorphic to the plane (that is, the affine space of dimension 2).

Using the Lemma and the Proposition, we may easily compute the zeta function of a non-degenerate cubic equation

$$a_1X^2 + a_2XY + a_3Y^2 + b_1X + b_2Y + c$$

in $\mathbb{A}^2$. (See the exercise below.)

Exercise 8.1. Let p be a prime. Compute the congruent zeta functions of the following two equations (varieties) over $\mathbb{F}_p$.

1. $V(X^2 + Y^2 - 1) \subset \mathbb{A}^2$.
2. $V(1 + Y^2) \subset \mathbb{A}^1$.
3. $V_h(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$.

Is there any relation between them? (Why?)

For the sake of completeness, we should have shown the following lemma.

LEMMA 8.4. *Let p be an odd prime. Let ζ be a primitive 8-th root of unity in $\overline{\mathbb{F}_p}$. That means, ζ is a root of $X^4 + 1 \in \mathbb{F}_p[X]$. Let us put $x = \zeta + \zeta^{-1}$. Then:*

(1) $x^2 = 2$.
(2) $x^p - x = 0$ *if* $p = \pm 1 \mod 8$.
(3) $x^p + x = 0$ *if* $p = \pm 3 \mod 8$.
(4) $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.

See the book of Serre (cited in No.01) for a proof.