CONGRUENT ZETA FUNCTIONS. NO.12

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In this lecture we make use of a scheme X of finite type over $\mathbb{Z}$, It is a patchwork of affine schemes of finite type over $\mathbb{Z}$,

An affine scheme X of finite type over $\mathbb{Z}$, in turn, is related to a set $f_1, \dots f_m$ of polynomial equations of coefficients in $\mathbb{Z}$, and is written as $\operatorname{Spec}(A)$ for a ring $A = \mathbb{Z}[X_1, \dots X_n]/(f_1, \dots f_m)$ of finite type over $\mathbb{Z}$, We consider quasi coherent sheaves over these objects. When X is affine ($X = \operatorname{Spec}(A)$), the category of quasi coherent sheaves over X is equivalent to the category of A-modules.

For any scheme X of finite type over $\mathbb{Z}$, we put

$$\zeta(X, s) = \prod_{x \in \operatorname{Spm}(X)} (1 - (Nx)^{-s})^{-1}$$

It is equal to the zeta function of the category of quasi coherent sheaves on X.

Recall we have defined the congruent zeta function as

$$Z(X, t) = \exp(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m).$$

Proposition 12.1. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then we have*

$$\zeta(X, s) = Z(X, q^{-s}).$$

PROOF.

$$\begin{aligned}
&\log(\zeta(X,s)) \\
&= -\sum_{\mathfrak{m}\in \operatorname{Spm}(X)} \log(1-N(\mathfrak{m})^{-s}) \\
&= \sum_{\mathfrak{m}} \sum_{r=1}^{\infty} \frac{N(\mathfrak{m})^{-rs}}{r} \\
&= \sum_{u=1}^{\infty} \sum_{\substack{\mathfrak{m} \\ [R/\mathfrak{m}:\mathbb{F}_q]=u}} \sum_{r=1}^{\infty} \frac{N(\mathfrak{m})^{-rs}}{r} \\
&= \sum_{u=1}^{\infty} \frac{\#X(\mathbb{F}_{q^u})_*}{u} \sum_{r=1}^{\infty} \frac{q^{-urs}}{r} = (\heartsuit).
\end{aligned}$$

where we put

$$X(\mathbb{F}_{q^u})_* = X(\mathbb{F}_{q^u}) \setminus \cup_{t<u}(X(\mathbb{F}_{q^t})).$$

Now let us put $t = q^{-s}$ and proceed further.

$$\begin{aligned}
(\heartsuit) &= \sum_{u=1}^{\infty}\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{q^u})_*}{ur} t^{ur} \\
&= \sum_{m} \frac{t^m}{m} \sum_{ur=m} \#X(\mathbb{F}_{q^u})_* \\
&= \sum_{m} \frac{t^m}{m} \#X(\mathbb{F}_{q^m}) \\
&= -\log(Z(X,t))
\end{aligned}$$

□

PROPOSITION 12.2. *Let X be a scheme of finite type over $\mathbb{Z}$. then we have*

$$\zeta(X,s) = \prod_{p:\text{prime}} \zeta((X \mod p), s).$$

Where we define $X \mod p$ as a fiber product $X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$.

LEMMA 12.3.

$$Z(X \times \mathbb{A}^1, t) = Z(X, qt).$$
$$\zeta(X \times \mathbb{A}^1, s) = \zeta(X, s-1)$$