三角化可能性(1) 固有ベクトルの存在によりサイズを1へらす.

$A \in M_n(\mathbb{C})$ があったとする.

λ : A の固有値.

v : 対応する固有ベクトル　$A u_1 = \lambda u_1$

$\parallel$

u_1

$\mathbb{C}^n$ の基底 $u_1, u_2, u_3, \cdots, u_n$ をもってくる.

$$A\underbrace{(u_1\ u_2 \cdots u_n)}_{\substack{\parallel \\ P \\ \text{正則}}} = (u_1 \cdots u_n)\left(\begin{array}{c|c} \lambda & \\ 0 & * \\ \vdots & \\ 0 & \end{array}\right)$$

$$= (u_1 \cdots u_n)\begin{array}{cc} \left(\begin{array}{c|c} \lambda & w \\ \hline 0 & B \end{array}\right) & \begin{array}{l} 1 \\ n-1 \end{array} \\ \begin{array}{cc} 1 & \quad n-1 \end{array} & \end{array}$$

$$AP = P\begin{pmatrix} \lambda & w \\ 0 & B \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \lambda & w \\ 0 & B \end{pmatrix}$$

$$A \sim \begin{pmatrix} \lambda & w \\ 0 & B \end{pmatrix}$$

n にかんする帰納法 → B が三角化可能

三角化可能性(2)へ示したあと

$Q^{-1}BQ = \underset{\text{三角行列}}{T}$ $\quad ({}^{\exists}Q \in GL_{n-1}(\mathbb{C})$
${}^{\exists}\underset{\text{三角}}{T} \in M_n(\mathbb{C}))$

$A \sim \begin{pmatrix} \lambda & w \\ 0 & B \end{pmatrix}$

$Q_1 = \begin{array}{c} \left(\begin{array}{c|c} 1 & 0 \\ 0 & Q \end{array}\right) \begin{array}{l} \}1 \\ \}n-1 \end{array} \\ \begin{array}{cc} 1 & n-1 \end{array} \end{array}$ とおく.

$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix}} \cdot Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} = 1_n$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}^{-1}}_{\begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix}} \begin{pmatrix} \lambda & w \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} \lambda & * \\ 0 & \underset{\underset{\text{三角}}{T}}{\underset{\shortparallel}{Q^{-1}BQ}} \end{pmatrix} \quad \text{三角}$$

$A \sim \begin{pmatrix} \lambda & w \\ 0 & B \end{pmatrix} \sim \begin{pmatrix} \lambda & * \\ 0 & T \end{pmatrix}$

$\square$

Cayley-Hamilton (1)

$A \in M_n(\mathbb{C}) \Rightarrow \Phi_A(A) = 0$

Aの固有多項式

◎ <u>lemma 10.3</u> $A \in M_n(\mathbb{C})$, $P \in GL_n(\mathbb{C})$

$\Rightarrow f(P^{-1}AP) = P^{-1}f(A)P$ (fは多項式)

(代入と共役は可換)

<u>証明</u>

$f(x) = X^2$のとき.

$f(P^{-1}AP) = P^{-1}A\cancel{P \cdot P^{-1}}AP = P^{-1}A^2P = P^{-1}f(A)P$

同様に $f(x) = X^n$のとき

$f(P^{-1}AP) = \underbrace{P^{-1}APP^{-1}AP \cdots P^{-1}AP}_{n回} = P^{-1}A^nP$

$= P^{-1}f(A)P$

♡

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Cayley-Hamilton (2) (相似な行列の固有多項式は等しい)

$A \in M_n(\mathbb{C})$ をとる.

$$A = PTP^{-1} \quad ({}^{\exists}P \in GL_n(\mathbb{C}))$$

$$\begin{aligned}
\Phi_A(x) &= \det(x1_n - A) \\
&= \det(x1_n - PTP^{-1}) \\
&= \det(P(x1_n - T)P^{-1}) \\
&= \det(P)\det(x1_n - T)(\det P)^{-1} \\
&= \det(x1_n - T) \\
&= \Phi_T(x)
\end{aligned}$$

$$\boxed{\det(AB) = \det(A)\det(B)}$$

$$\Phi_A(x) = \Phi_T(x) \quad (T = P^{-1}AP) \quad (A = PTP^{-1})$$

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Cayley-Hamilton (3) (三角行列への帰着)

$$A = PTP^{-1}$$

$$\Phi_A(x) = \Phi_T(x)$$

$$\Phi_A(A) = \Phi_T(A) = \Phi_T(PTP^{-1})$$

$$= P\,\Phi_T(T)\,P^{-1}$$

$$\Phi_A(A) \sim \Phi_T(T)$$

→ Aが三角行列のときにCHを示せばよい.

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CH(4) (三角行列のとき)

$A = \begin{pmatrix} \lambda & * \\ 0 & B \end{pmatrix}$ のとき

$$\Phi_A(x) = \det(x 1_n - A)$$

$$= \det \left(\begin{array}{c|c} x-\lambda & * \\ \hline O & x 1_{n-1} - B \end{array} \right)$$

$$= (x-\lambda) \det(x 1_{n-1} - B)$$

$$= (x-\lambda) \Phi_B(x)$$

$1_n = \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix}$

$\Rightarrow$ $A = \begin{pmatrix} \lambda & * \\ 0 & B \end{pmatrix}$ のとき $\Phi_A(x) = (x-\lambda)\Phi_B(x)$

このとき

$$\Phi_A(A) = (\cancel{x}A - \lambda 1_n) \Phi_B(A)$$

$$\Phi_B(A) = \begin{pmatrix} * & * \\ 0 & \Phi_B(B) \end{pmatrix}, \quad \Phi_B(B) = 0 \quad \leftarrow 帰納法$$

$$= \begin{pmatrix} 0 & * \\ O & * \end{pmatrix} \begin{pmatrix} * & * \\ O & O \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$\therefore \Phi_A(A) = O$$

□

和 11.17(2)

$\cancel{B=}$

$$\underbrace{B\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 1 \end{pmatrix}}_{P} = \begin{pmatrix} 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} & 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} & 2\begin{pmatrix} 7 \\ 8 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 9\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 9 \\ 0 & 0 & 2 \end{pmatrix}}_{\text{三角}}$$