CONGRUENT ZETA FUNCTIONS. NO.11

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Let f be a self map $f : M \to M$ of a set M. It defines a (discrete) **dynamical system** (M, f).

To explain the basic idea, we first examine the case where M is a finite set.

We put $A = C(M, \mathbb{C})$, the set of $\mathbb{C}$-valued functions on M.

f defines a pull-back of functions:

$$f^*(a)(x) = a(f(x)) \qquad (a \in A)$$

and push-forward:

$$f_*(a)(x) = \sum_{f(y)=x} a(y) \qquad (a \in A).$$

(It might be better to treat the push-forward as above as a push-forward of measures.)

We note also that any element of A admits an integration

$$\int_M a = \sum_{x \in M} a(x) \qquad (a \in A)$$

(which is a integration with respect to the counting measure.)

Proposition 11.1. *We have*

$$\int_M (f^* a) b = \int_M a(f_* b)$$

In other words, f_* is the adjoint of f^*.

Proposition 11.2. *Let us put $M = \{1, 2, \dots, n\}$. Let $e_1, \dots, e_n$ be the indicators of elements of M. Then $\{e_1, \dots e_n\}$ forms a basis of A. f^* is represented by a matrix $P_f = (\delta_{f(i)j})$. f_* is represented by a matrix ${}^tP_f = (\delta_{if(j)})$.*

Definition 11.3. We define the set $\mathrm{Fix}(f)$ as the set of fixed points of f. Namely,

$$\mathrm{Fix}(f) = \{x \in M; f(x) = x\}.$$

Proposition 11.4. $\mathrm{tr}(f^*) = \mathrm{tr}(f_*) = \# \mathrm{Fix}(f)$.

It should be noted that $\mathrm{tr}((f^k)^*)$ may be comuted using a "path-integral"-like formula.

$$\mathrm{tr}((f^k)^*) = \sum_{\alpha \in M^k} P_{\alpha_1 \alpha_2} P_{\alpha_2 \alpha_3} \dots P_{\alpha_{k-1} \alpha_k} P_{\alpha_k \alpha_1}$$

Definition 11.5. We define the Artin-Mazur zeta function of a dynamical system (M, f) as

$$Z((M, f), T) = \exp(\sum_{j=1}^{\infty} \frac{\# \mathrm{Fix}(f^j) T^j}{j})$$

Proposition 11.6.

$$Z((M, f), T) = \frac{1}{\det(1 - T f^*)}$$

11.1. **Congruent zeta as a zeta of a dynamical system.** The definition of Artin Mazur zeta function is valid without assuming the number of the base space M to be a finite set.

Definition 11.7. Let M be a set. Let $f : M \to M$ be a map such that $\# \operatorname{Fix}(f^n)$ is finite for any $n > 0$. We define the Artin-Mazur zeta function of a dynamical system (M, f) as

$$Z((M,f),T) = \exp(\sum_{j=1}^{\infty} \frac{\# \operatorname{Fix}(f^j) T^j}{j})$$

Let q be a power of a prime p. We may consider an automorphism Frob_q of $\bar{\mathbb{F}}_q$ over $\mathbb{F}_q$ by

$$\operatorname{Frob}_q(x) = x^q$$

Proposition 11.8. $\operatorname{Frob}_q : \mathbb{F}_{q^r} \to \mathbb{F}_{q^r}$ *is an automorphism of order* r*. It is a generator of the Galois group* $\operatorname{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$*.*

For any projective variety X defined over $\mathbb{F}_q$, we may define a Frobenius action Frob_q on $X(\bar{\mathbb{F}}_q)$:

$$\operatorname{Frob}_q([x_0 : x_1 : \dots x_N]) = ([x_0^q : x_1^q : \dots x_N^q]).$$

For any $\bar{\mathbb{F}}_q$-valued point $x \in X(\bar{\mathbb{F}}_q)$, We have

$$\operatorname{Frob}_q^r(x) = x \iff x \in X(\mathbb{F}_{q^r}).$$

Proposition 11.9. *The Artin Mazur zeta function of the dynamical system* $(X(\bar{\mathbb{F}}_q), \operatorname{Frob}_q)$ *conincides with the congruent zeta function* $Z(X/\mathbb{F}_q, t)$*.*