# CONGRUENT ZETA FUNCTIONS. NO.5

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For any projecvite variety $V$ over a field $\mathbb{F}_q$, we may define its congruent zeta function $Z(V/\mathbb{F}_q, T)$ likewise for the affine varieties.

We quote the famous Weil conjecture

CONJECTURE 5.1 (Now a theorem [1] ). Let $X$ be a projective smooth variety of dimension $d$. Then:

(1) (Rationality) There exists polynomials $\{P_i\}$ such that
$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)\dots P_{2d-1}(X,T)}{P_0(X,T)P_2(X,T)\dots P_{2d}(X,T)}.$$
(2) (Integrality) $P_0(X,T) = 1-T$, $P_{2d}(X,T) = 1-q^dT$, and for each $r$, $P_r$ is a polynomial in $\mathbb{Z}[T]$ which is factorized as
$$P_r(X,T) = \prod(1-a_{r,i}T)$$
where $a_{r,i}$ are algebraic integers.
(3) (Functional Equation)
$$Z(X, \frac{1}{q^dT}) = \pm q^{\frac{d\chi}{2}} T^\chi Z(t)$$
where $\chi = (\Delta.\Delta)$ is an integer.
(4) (Rieman Hypothesis) each $a_{r,i}$ and its conjugates have absolute value $q^{r/2}$.
(5) If $X$ is the specialization of a smooth projective variety $X$ over a number field, then the degeee of $P_r(X,T)$ is equal to the $r$-th Betti number of the complex manifold $X(\mathbb{C})$. (When this is the case, the number $\chi$ above is equal to the "Euler characteristic" $\chi = \sum_i (-1)^i b_i$ of $X(\mathbb{C})$.)

It is a profound theorem, relating the number of rational points $X(\mathbb{F}_q)$ of $X$ over finite fields and the topology of $X(\mathbb{C})$.

For a further study we recommend [?, Appendix C],[?], [?].

projective space and projective varieties.

DEFINITION 5.2. Let $R$ be a ring. A polynomial $f(X_0, X_1, \dots, X_n) \in R[X_0, X_1, \dots, X_n]$ is said to be **homogenius** of degree $d$ if an equality
$$f(\lambda X_0, \lambda X_1, \dots, \lambda X_n) = \lambda^d f(X_0, X_1, \dots, X_n)$$
holds as a polynomial in $n+2$ variables $X_0, X_1, X_2, \dots, X_n, \lambda$.

DEFINITION 5.3. Let $k$ be a field.

(1) We put
$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times$$
and call it (the set of $k$-valued points of) the **projective space**. The class of an element $(x_0, x_1, \dots, x_n)$ in $\mathbb{P}^n(k)$ is denoted by $[x_0 : x_1 : \dots : x_n]$.

[1] There are a lot of people who contributed. See the references.

(2) Let $f_1, f_2, \dots, f_l \in k[X_0, \dots, X_n]$ be homogenious polynomials. Then we set

$$V_h(f_1, \dots, f_l) = \{[x_0 : x_1 : x_2 : \dots x_n]; f_j(x_0, x_1, x_2, \dots, x_n) = 0 \qquad (j = 1, 2, 3, \dots, l)\}.$$

and call it (the set of $k$-valued point of) the **projective variety** defined by $\{f_1, f_2, \dots, f_l\}$.

(Note that the condition $f_j(x) = 0$ does not depend on the choice of the representative $x \in k^{n+1}$ of $[x] \in \mathbb{P}^n(k)$.)

LEMMA 5.4. *We have the following picture of* $\mathbb{P}^2$.

(1)
$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1.$$
*That means,* $\mathbb{P}^2$ *is divided into two pieces* $\{Z \neq 0\} = \complement V_h(Z)$ *a nd* $V_h(Z)$.

(2)
$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$
*That means,* $\mathbb{P}^2$ *is covered by three "open sets"* $\{Z \neq 0\}, \{Y \neq 0\}, \{X \neq 0\}$. *Each of them is isomorphic to the plane (that is, the affine space of dimension 2).*

## 5.1. **proj.**

DEFINITION 5.5. An $\mathbb{N}$-graded ring $S$ is a commutative ring with a direct sum decomposition

$$S = \bigoplus_{i \in \mathbb{N}} S_i \qquad \text{(as a module)}$$

such that $S_i S_j \subset S_{i+j} \quad (\forall i, j \in \mathbb{N})$ holds. We define its irrelevant ideal $S_+$ as

$$S_+ = \bigoplus_{i>0} S_i.$$

An element $f$ of $S$ is said to be homogenous if it is an element of $\cup S_i$. An ideal of $S$ is said to be homogeneous if it is generated by homogeneous elements. Homogeneous subalgebras are defined in a same way.

DEFINITION 5.6.

$$\mathrm{Proj}(S) = \{\mathfrak{p}; \mathfrak{p} \text{ is a homogeneous prime ideal of } S, \mathfrak{p} \not\supset S_+\}$$

For any homogeneous element $f$ of $S$, we define a subset $D_f$ of $\mathrm{Proj}(S)$ as

$$D_f = \{\mathfrak{p} \in \mathrm{Proj}\, S; \mathfrak{p} \not\ni f\}.$$

$\mathrm{Proj}\, S$ has a topology (Zariski topology) which is defined by employing $\{D(f)\}$ as an open base.

PROPOSITION 5.7. *For any graded ring $S$ and its homogeneous element $f$, $S[\frac{1}{f}]$ also carries a structure of graded ring. There is a homeomorphism*

$$D_f \sim \mathrm{Spec}(S[\frac{1}{f}])_0).$$

*We may define, via these homeo altogether, a locally ringed space structure on* $\mathrm{Proj}(S)$.