No.05.1: ring of Witt vectors (1) Preparations:suppliment

We need the following lemma. (Note that for any $f, g \in 1+TA[[T]]$, we have by definition $(f)_W + (g)_W = (fg)_W$).

Lemma 5.1. *Let A be any commutative ring. Then every element $(f)_W$ of $\Lambda(A) = 1 + TA[[T]]$ is written uniquely as*

$$(f)_W = \sum_{j=1}^{\infty} (1 - x_j T^j)_W \qquad (x_j \in A).$$

Proof. Let us prove this in induction. Assume we already have $x_1, \dots, x_n \in A$ such that

$$(f)_W - \sum_{j=1}^{n} (1 - x_j T^j)_W = (1 + \text{ (terms of order higher than } n+1))_W.$$

That means,

$$(f)_W - \sum_{j=1}^{n} (1 - x_j T^j)_W = (1 - aT^{n+1} - bT^{n+2} + \dots)_W \quad (\exists a, b, \dots \in A)$$

Now, let us put $x_{n+1} = a$. We then compute and see:

$$\begin{aligned} (f)_W - \sum_{j=1}^{n+1} (1 - x_j T^j)_W &= (1 - aT^{n+1} - bT^{n+2} + \dots)_W - (1 - aT^{n+1})_W \\ &= ((1 - aT^{n+1})^{-1}(1 - aT^{n+1} - bT^{n+2} + \dots))_W \\ &= (1 + (\text{terms of order higher than } n+2))_W \end{aligned}$$

And that's it. □

Corollary 5.2. $\Lambda(A) = 1 + TA[[T]]$ *is generated by*

$$\{(1 - x_j T^j)_W; \quad x_j \in A, \quad j = 1, 2, 3, \dots\}$$

as a topological module.