$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.05: ring of Witt vectors (1) Preparations

From here on, we make use of several notions of category theory. Readers who are unfamiliar with the subject is advised to see a book such as [?] for basic definitions and first properties.

Let p be a prime number. For any commutative ring k of characteristic $p \neq 0$, we want to construct a ring $W(k)$ of characteristic 0 in such a way that:

(1) $W(\mathbb{F}_p) = \mathbb{Z}_p$.
(2) $W(\bullet)$ is a functor. That means,
 (a) For any ring homomorphism $\varphi : k_1 \to k_2$ between rings of characterisic p, there is given a unique ring homomorphism $W(\varphi) : W(k_1) \to W(k_2)$.
 (b) $W(\bullet)$ should furthermore commutes with compositions of homomorphisms.

Recent days, it gets easier for us on the net to i find some good articles concerning the ring of Witt vectors. The treatment here borrows some ideas from them. See for example the "comments" section in `https://www.encyclopediaofmath.org/index.php/Witt_vector`

5.1. $\Lambda(A)$.

Definition 5.1. For any commutative ring A,

(1) we define

$$\Lambda(A) = (1 + TA[[T]]) \qquad \text{(as a set)}$$

For any $f \in (1+TA[[T]])$, we denote by $(f)_W$ the corresponding element in $\Lambda(A)$.

(2) For any $(f)_W$, $(g)_W \in \Lambda(A)$, we define their sum by

$$(f)_W + (g)_W = (fg)_W$$

It is easy to see that $\Lambda(A)$ is an additive group. It also carries the "T-addic topology" so that $\Lambda(A)$ is a topological additive group.

The next task is to define multiplicative structure on $\Lambda(A)$. To that end, we do something somewhat different to others.

Definition 5.2. For any commutative ring A, we define $E(A) = \mathrm{End}_{\mathrm{additive}}(\Lambda(A))$. It has the usual structure of a ring. For any $a \in A$, we define its "Teichmüler" lift $[a]$ as

$$(f(T))_W \mapsto (f(aT))_W.$$

The basic idea is to define $E_0(A)$ as the subalgebra of $E(A)$ topological-algebraically generated by all the Teichmüller lifts $\{[a]; a \in A\}$ and identify $E_0(A)$ with $\Lambda(A)$. To avoid some difficulties doing so, we first do this when A is a very good one:

Proposition 5.3. *Assume $A = \Omega$, an algebraically closed field. Then:*

(1) $\Lambda(A)$ *is generated by* $\{(1-aT)_W | a \in A\}$ *as a topological additive group.*

(2) *The subring $E_0(A)$ of $E(A)$ generated by $\{[a]|a \in A\}$ as a topological ring is equal to $\{x \in E(A); x$ commutes with all Teichmüller lifts$\}$.*

(3) *$(1-T)_W$ is a generating separating vector of $\Lambda(A)$ over $E_0(A)$. Thus we have a module isomorphism*

$$E_0(A) \ni \varphi \to \varphi((1-T)_W) \in \Lambda(A).$$

(Note that This isomorphism sends $[a]$ to $(1-aT)_W$.

We may thus identify $E_0(A)$ and $\Lambda(A)$ via this isomorphism and employ a ring structure on $\Lambda(A)$.

Here after, for any algebraically closed field A, we employ the ring structure of $\Lambda(A)$ defined as the above proposition. In this language we have:

$$(1-aT)_W \cdot (1-bT)_W = (1-abT)_W \qquad (a, b \in A)$$

More generally, for any $f(T) \in 1 + TA[[T]]$, we have a formula for multiplication by degree-1-object $(1-aT)_W$:

$$(1-aT)_W \cdot (f(T))_W = (f(aT)_W) \qquad (a \in A)$$

We may extend this formula to any polynomial $g(T) \in 1 + TA[T]$ with constant term=1. Indeed, we factorize g as $g(T) = \prod_{j=1}^{k}(1-\alpha_j T)$ and

$$(g(T))_W \cdot (f(T))_W = \prod_j f(\alpha_j T)$$

Exercise 5.1. Compute $(1+aT+bT^2)_W(1+pT+qT^2)_W$. Notice that the result of the computation only needs polynomials with coefficents in $\mathbb{Z}[a,b,p,q]$ rather than some extension of the ring.