$\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.9: The ring of Witt vectors and $\mathbb{Z}_p$

Definition 9.1. Let A be a ring of characteristic p. We call the ring

$$\Lambda_n^{(p)}(A) = \Lambda^{(p)}(A)/V_p^n(\Lambda^{(p)}(A))$$

the ring of Witt vectors of length n. Its elements are called **Witt vectors of lenth** n.

Note that

$$\Lambda^{(p)}(A)/V_p^n(\Lambda^{(p)}(A))$$

may be considered as a set A^n with an unusual ring structure.

Proposition 9.2.

$$\mathbb{Z}_p \cong \Lambda^{(p)}(\mathbb{F}_p).$$

Proof. Since $\Lambda^{(p)}$ is a unital commutative ring, there naturally exists a natural ring homomorphism

$$\iota : \mathbb{Z} \to \Lambda^{(p)}(\mathbb{F}_p).$$

Let us first fix a positive integer n and examine the kernel K_n of a map

$$\pi_n \circ \iota : \mathbb{Z} \to \Lambda^{(p)}(\mathbb{F}_p)/V_p^n(\Lambda^{(p)}(\mathbb{F}_p))$$

where π_n is the natural projection. Since

$$\#\Lambda^{(p)}(\mathbb{F}_p)/V_p^n(\Lambda^{(p)}(\mathbb{F}_p)) = \#(\mathbb{F}_p^n) = p^n,$$

we have

$$\#(\mathbb{Z}/K_n)|p^n.$$

In other words, $K_n = p^s\mathbb{Z}$ for some integer s. On the other hand, we have

$$\pi_n \circ \iota(p^s) = (1-T)^{p^s} = (1-T^{p^s})$$

thus

$$\pi_n \circ \iota(p^s) \in K_n \iff s \geq n.$$

This implies that $K_n = p^n\mathbb{Z}$ and therefore we have an inclusion

$$\mathbb{Z}/p^n\mathbb{Z} \hookrightarrow \Lambda_n^{(p)}(\mathbb{F}_p)$$

which turns to be a bijection ($\because \#(\mathbb{Z}/p^n\mathbb{Z}) = \#(\Lambda_n^{(p)}(\mathbb{F}_p))$).

We then take a projective limit of the both hand sides and obtain the resired isomorphism.

□