NORM BASED EXTENSION OF REFLEXIVE MODULES OVER WEYL ALGEBRAS

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Abstract. We consider a reflexive module of rank one over a degenerate Weyl algebra over a field of positive characteristic. We define an invariant which we call wrinkle of the module and see that it is good enough to distinguish trivial module.

Introduction

Let $\Bbbk$ be a field. Let $A_{n,m}(\Bbbk)$ be a degenerate Weyl algebra with m central generators along with $2n$ generators which satisfy canonical commutation relations. We study reflexive modules of rank one over $A_{n,m}(\Bbbk)$.

When $n = 0$, then $A_{0,m}$ is an ordinary polynomial algebra of m variables. Since the polynomial algebra is a unique factorization domain, it is well known that such module is always trivial in this case (see for example [7, Chapter VII, Section 3]).

In contrast, when $n \geq 1$, there are (infinitely) many reflexive $A_{n,m}(\Bbbk)$-modules of rank one. For example, it is known that there are infinitely many $A_{1,0}$-modules. They are parametrized by "Calogero-Moser spaces" (see for example [5]).

In this paper, we concentrate on the case where the characteristic p of the base field $\Bbbk$ is non-zero and give an invariant for each reflexive $A_{n,m}$-module W of rank one.

To that aim, we use the fact that the algebra $A_{n,m}$ has a large subalgebra Z in its center and that W is associated with a reflexive sheaf W_X on $X = \operatorname{Spec}(Z) = \mathbb{A}^{2n+m}$.

We then consider a reflexive extension $W_{\bar{X}}$ of the reflexive sheaf W_X on $X = \mathbb{A}^{2n+m}$ to the projective space $\bar{X} = \mathbb{P}^{2n+m}$. The extension $W_{\bar{X}}$ has some "wrinkles" at infinity, that means, the locus in the hyperplane H at infinity where $W_{\bar{X}}$ is not locally free.

The extension of the algebra sheaf A_X to $\bar{X}$ is defined in Definition 2.3. The extension of W_X to $\bar{X}$ is then is defined in Definition 3.7. The definition uses some properties of "norms" on W which are stated in Subsection 3.1.

We prove in Theorem 4.1 that when $\dim X (= 2n + m) \geq 3$, the wrinkle gives an invariant good enough to distinguish a trivial one. Namely, if the reflexive extension has no "wrinkles" at infinity, then the module W is trivial. The proof depends on a result

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proved by Abe and Yoshinaga [1] which is a generalization of Horrocks' theorem to the reflexive sheaves.

We note that the only case where the condition $2n + m \geq 3$ is not satisfied (except for the above-discussed case where $n = 0$) is the case $n = 1, m = 0$. In that case, the wrinkle is void for any W because any reflexive sheaf on a regular scheme of dimension 2 is always locally free [10]. Even then one may always achieve the condition by adding some extra variables. We deal with such an example in Subsection 3.5

The idea of extending projective A-module to a "sheaf" on a "quantum projective space" already appears in the literature (see [6] for example). Our approach here deals objects over a field of non-zero characteristics. We may then obtain results on corresponding objects over a field of characteristic zero by using a technique of ultrafilter as discussed for example in [13] and [14]. The benefit of our approach is that we are able to use usual theory and techniques of algebraic geometry directly rather than to develop the "q-analogues" of them.

Let us describe the motivation of the present paper. There are two conjectures in our mind. One is Dixmier's conjecture which states that every endomorphism of Weyl algebra is an automorphism. The other is the Jacobian problem which states that every polynomial map of the affine space $\mathbb{A}^n$ with the constant Jacobian is actually invertible. (There are several conjectures that are equivalent or deeply related to the above two conjectures. See for example [8], [2].) The two conjectures are stably equivalent [15], [13], [4]. Indeed, it is shown in [13, Proposition 7.1] that there exists a map

$$\mathcal{L} : \mathrm{End}^{\mathrm{alg}}_{\mathbb{C}}(A_n(\mathbb{C})) \ni \varphi \mapsto f_\varphi \in \mathrm{End}_1(\mathbb{A}^{2n}(\mathbb{C}))$$

which associates an endomorphism of a Weyl algebra $A_n(\mathbb{C})$ to a symplectic polynomial map of $\mathbb{A}^{2n}(\mathbb{C})$. The strategy for giving such map is to regard the field $\mathbb{C}$ and schemes on it as a "ultra filter limit" of fields and schemes of characteristic $p \neq 0$.

It is fairly easy to see that the correspondence $\mathcal{L}$ is injective, for example by using [12, Lemma 11]. We may ask if it is surjective, in other words, if every symplectic map f of $\mathbb{A}^{2n}(\mathbb{C})$ is liftable to an endomorphism φ of $A_n(\mathbb{C})$ [3]. According to our strategy the problem is equivalent to asking if the liftability is true if we replace the base field $\mathbb{C}$ by a base field $\mathbb{k}$ of characteristic p and assume that degree of f is sufficiently smaller than p. The liftability is then interpreted, by using a theory of connections, as a problem of triviality of an $A_n(\mathbb{k})$-module $W^{(f)}$ defined by f [3], [16].

On the other hand, it is already known that elementary morphisms are liftable. Since any automorphism of $\mathbb{A}^2$ is elementary, any automorphism of $\mathbb{A}^2$ is liftable. Thus we may imagine in general that the triviality of $W^{(f)}$ somehow measures the obstruction of f to be invertible. So we expect that the study of $W^{(f)}$ may give some clue to the two conjectures mentioned above.

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1. Notations

Definition 1.1. Let $\Bbbk$ be a field. Let n, m be non-negative integers. A *degenerate Weyl algebra* $A_{n,m}(\Bbbk)$ over $\Bbbk$ is a tensor product of ordinary Weyl algebra $A_n(\Bbbk)$ and an ordinary polynomial algebra in m variables. In other words, $A_{n,m}$ is an algebra over $\Bbbk$ generated by $2n+m$ elements, which we call *standard generators*,

$$\{\gamma_1, \gamma_2, \ldots, \gamma_{2n}, \gamma_{2n+1}, \gamma_{2n+2}, \ldots, \gamma_{2n+m}\}$$

with $\gamma_{2n+1}, \gamma_{2n+2}, \ldots, \gamma_{2n+m}$ being central and other γ's satisfy the "canonical commutation relations". Namely, they satisfy the equations

$$[\gamma_i, \gamma_j](= \gamma_i\gamma_j - \gamma_j\gamma_i) = h_{ij} \qquad (1 \le i, j \le 2n+m), \tag{CCR}$$

where h is an anti-Hermitian $(2n+m) \times (2n+m)$ matrix of the form

$$(h_{ij}) = \begin{pmatrix} 0 & -1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 0_m \end{pmatrix}.$$

Note that we include the case $m = 0$ where the degenerate Weyl algebra $\mathbb{A}_{n,0}(\Bbbk) = \mathbb{A}_n(\Bbbk)$ is in practice "non-degenerate".

Definition 1.2. Let $A = A_{n,m}(\Bbbk)$ be the degenerate Weyl algebra over a field $\Bbbk$ of characteristic p. We put

$$\begin{aligned} Z &= \Bbbk[t_1^p, t_2^p, \ldots, t_{2n}^p, t_{2n+1}^p, \ldots, t_{2n+m}^p], \\ C &= \Bbbk[t_1^p, t_2^p, \ldots, t_{2n}^p, t_{2n+1}, \ldots, t_{2n+m}], \\ S &= \Bbbk[t_1, t_2, \ldots, t_{2n}, t_{2n+1}, \ldots, t_{2n+m}] \end{aligned}$$

and identify C with the center of A by introducing the relations

$$\begin{aligned} t_j^p &= \gamma_j^p \qquad (j = 1, 2, 3, \ldots, 2n), \\ t_j &= \gamma_j \qquad (j = 2n+1, 2n+2, \ldots, 2n+m). \end{aligned}$$

We further put $K = Q(Z)$, $L = Q(S)$ the quotient fields of the above rings Z, S, respectively.

The subalgebra Z certainly depends on the choice of the standard generators $\{\gamma_j\}$. That means, it is not invariant under algebra automorphisms of A. It seems possible that Z is invariant under "small" automorphisms of A, that means, compositions of automorphisms of low degrees.

We have a standard matrix representation

$$\Phi_0 : A \to M_{p^n}(S)$$

which is explained by using extra indeterminates $x_1, x_2, \dots, x_n$ as follows. We consider a free S-module

$$V = S[x_1, x_2, \dots, x_n]/(x_1^p, \dots, x_n^p).$$

Φ_0 may then be written in the following manner:

$$\begin{cases} \Phi_0(\gamma_j) = t_j + x_j & (j = 1, 2, \dots, n), \\ \Phi_0(\gamma_j) = t_j + \dfrac{\partial}{\partial x_{j-n}} & (j = n+1, n+2, \dots, 2n), \\ \Phi_0(t_j) = t_j & (j = 2n+1, 2n+2, \dots, 2n+m). \end{cases}$$

It is easy to verify that Φ_0 is indeed a faithful representation of the $\Bbbk$-algebra A. We may describe the representation Φ_0, as we have done in author's papers [12, 13, 14, 15, 16], by introducing matrices $\mu_1, \mu_2, \dots, \mu_n, \nu_1, \nu_2, \dots, \nu_n \in M_{p^n}(\Bbbk)$ acting on

$$\Bbbk^{p^n} \cong \Bbbk[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$$

as $\mu_j = x_j$, $\nu_j = \partial/\partial x_j$. Namely Φ_0 may be written as follows.

$$\text{(AB1)} \qquad \begin{cases} \Phi_0(\gamma_j) = t_j + \mu_j & (j = 1, 2, \dots, n), \\ \Phi_0(\gamma_j) = t_j + \nu_{n-j} & (j = n+1, n+2, \dots, 2n), \\ \Phi_0(t_j) = t_j & (j = 2n+1, 2n+2, \dots, 2n+m). \end{cases}$$

This description also shows that $\Phi_0(\gamma_j)$ is asymptotically equal to a constant matrix t_j when we consider its behavior at infinity. The equation later generalizes to an equation (AB) and becomes a key in the proof of Lemma 2.1.

We define the *norm* of an element $s \in A$ by

$$N_A(s) = \det(\Phi_0(s))(\in S).$$

Finally we employ the symbol

$$\mathbb{N}_p = \{0, 1, 2, 3, \dots, p-1\}.$$

The author admits that the notation is a bit strange, but it may be helpful for writing down some cumbersome index sets such as

$$\mathbb{N}_p^n = \{0, 1, 2, 3, \dots, p-1\}^n$$

and so on.

2. The sheaf $A_{\bar{X}}$ on a projective space $\bar{X}$

2.1. The affine space X and its completion $\bar{X}$. We denote by X the $(2n+m)$-dimensional affine space

$$X = \mathrm{Spec}(Z) = \mathrm{Spec}(\Bbbk[t_1^p, t_2^p, \dots, t_{2n+m}^p])(\cong \mathbb{A}^{2n+m}).$$

We complete X to a projective space

$$\bar{X} = \mathrm{Proj}(\Bbbk[T_0^p, T_1^p, \dots, T_{2n+m}^p])(\cong \mathbb{P}^{2n+m}),$$

where homogeneous coordinates $\{T_j^p\}$ are related to the affine coordinates $\{t_j^p\}$ by

$$t_j = T_j/T_0 \qquad (j = 1, 2, 3, \dots, 2n+m).$$

We denote by $H = \partial X$ the hyperplane at infinity. Namely, it is a closed subscheme of $\bar{X}$ defined by a homogeneous ideal (T_0^p):

$$H = V_{\bar{X}}(T_0^p) = \mathrm{Proj}(\Bbbk[T_1^p, \dots, T_{2n+m}^p]).$$

In what follows, the dagger sign $(\bullet^\dagger)$ denotes the "inverse Frobenius pull-back" relative to $\Bbbk$. Namely, we use the following notation.

$$\begin{aligned}
X^\dagger &= \mathrm{Spec}(S) = \mathrm{Spec}(\Bbbk[t_1, t_2, \dots, t_{2n+m}]),\\
\bar{X}^\dagger &= \mathrm{Proj}(\Bbbk[T_0, T_1, \dots, T_{2n+m}]),\\
H^\dagger &= V_{\bar{X}^\dagger}(T_0) = \mathrm{Proj}(\Bbbk[T_1, \dots, T_{2n+m}]).
\end{aligned}$$

The following diagram is commutative and rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & I_H \arrow[r] \arrow[d] & \mathcal{O}_{\bar{X}} \arrow[r] \arrow[d] & \mathcal{O}_H \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & I_{H^\dagger} \arrow[r] & \mathcal{O}_{\bar{X}^\dagger} \arrow[r] & \mathcal{O}_{H^\dagger} \arrow[r] & 0
\end{tikzcd}$$

We should make a clear distinction between I_H and $I_{H^\dagger}$. They are related by the equation

$$I_H \cdot \mathcal{O}_{\bar{X}^\dagger} = I_{H^\dagger}^p. \tag{I}$$

Later we encounter a similar situation at Proposition 2.8 (2).

2.2. Definition of the norm-based extension $A_{\bar{X}}$. We denote by A_X the sheaf of algebras on X associated to the Z-algebra A. The norm map N_A defines a morphism

$$N_A : A_X \to \mathcal{O}_{X^\dagger}$$

of sheaves over X. In this subsection we define a sheaf $A_{\bar{X}}$ on $\bar{X}$ which extends A_X. We do so by using the idea that elements of A are "asymptotically commutative at infinity" so that the norm map tend to an algebra homomorphism as we approach to the infinity. To see such behavior, we define $\mathrm{ord}_{H^\dagger}(s)$ of an element $s \in L$ as the order of s with respect to the divisor $H^\dagger$. Also for any element a of A, let us denote by $\deg(a)$ the total degree of a with respect to the standard generators $\{\gamma_1, \dots, \gamma_{2n}, t_{2n+1}, \dots, t_{2n+m}\}$. They are related in the following manner.

Lemma 2.1. *Let $s = ab^{-1} (a, b \in A)$ be an element of $A \otimes_Z K$. Then we have*

$$\operatorname{ord}_{H^\dagger}(N_A(s)) = -p^n(\deg(a) - \deg(b)).$$

Proof. Let us write a in terms of standard generators $\{\gamma\}$ as

$$a = \sum_J a_J \gamma^J.$$

Then the matrix $\Phi_0(a)$ is of the following form:

$$\Phi_0(a) = \sum_{|J|=\deg(a)} a_J t^J + (\text{lower order terms in } \{t\} \text{ with matrix coefficients}). \tag{AB}$$

By taking the determinants of the both hand sides we thus see that $\operatorname{ord}_{H^\dagger}(N_A(a)) = -p^n(\deg(a))$ holds. The same argument is applied to the element $b \in A$. □

The equation (AB) above says that the matrix $\Phi_0(a)$ associated to an element $a \in A$ is asymptotically equal to a scalar matrix valued function $\sum_{|J|=\deg(a)} a_J t^J$. Thus the question of regularity of a section s of A_X at infinity may be controlled by the norm $N_A(s)$. To be more precise, we have the following lemma.

Lemma 2.2. *Let U be an affine open subset of $\bar{X}$ such that $U \cap H \neq \emptyset$. For any non-zero section s of $A_X(U \cap X)$, the following conditions are equivalent*:

1. $N_A(s) \in \mathcal{O}_{\bar{X}^\dagger}(U)$.
2. $N_A(s)$ *is regular on the generic point of* $H^\dagger$.
3. $\operatorname{ord}_{H^\dagger}(N_A(s)) \geq 0$.
4. *There exist* $a, b \in A$ *such that* $s = a/b$ *with* $\deg(a) \leq \deg(b)$.
5. s *is integral over the ring* $R = \mathcal{O}_X(U)$. *That means,* $R[s]$ *is an* R*-module of finite type.*
6. $\Phi_0(s) \in M_{p^n}(\mathcal{O}_{\bar{X}^\dagger})$.

Proof. (1) $\implies$ (2) and (2) $\iff$ (3) are trivial.
(2) $\implies$ (1): $N_A(s)$ is regular on U except for a subscheme of codimension at least 2 in U. Thus (1) follows from the fact that $\bar{X}$ is a normal scheme.
(4) $\iff$ (3) is a consequence of Lemma 2.1.
(6) $\implies$ (5): Cayley-Hamilton theorem.
(5) $\implies$ (1): The eigen values of s are integral over $\mathcal{O}_{\bar{X}}(U)$.
(4) $\implies$ (6) follows from an calculation similar to the one in the proof of Lemma 2.1. We see that the element $\Phi_0(s)$ is regular on $U \cap X$ and at the generic point of H. Thus we see that $\Phi_0(s)$ is regular on U except at a subscheme of codimension at least 2. Since U is a normal scheme, we conclude that $\Phi_0(s)$ is regular on the whole U. □

DEFINITION 2.3. Let us define a sheaf $A_{\bar{X}}$ by putting

$$A_{\bar{X}}(U) = \{f \in A \otimes_Z \mathcal{O}(U \cap X); \quad N_A(f) \in \mathcal{O}_{X^\dagger}(U)\}$$

for any open subset U of $\bar{X}$. Note that the condition of the right-hand side may be replaced by one (hence any) of the condition in Lemma 2.2. We call the sheaf $A_{\bar{X}}$ the *norm-based extension* of A.

PROPOSITION 2.4. *The norm-based extension $A_{\bar{X}}$ of A has the following properties.*

(1) *We may extend $\Phi_0 : A \to M_{p^n}(S)$ to a morphism of sheaves*

$$\Phi : A_{\bar{X}} \to M_{p^n}(\mathcal{O}_{\bar{X}^\dagger})$$

in such a way that an equation

$$\Phi(A_{\bar{X}}(U)) = \{f \in M_{p^n}(\mathcal{O}_{\bar{X}^\dagger}(U)); f|_{U \cap X} \in A(U \cap X)\}$$

holds for any open set U of $\bar{X}$.

(2) *The sheaf $A_{\bar{X}}$ is actually an $\mathcal{O}_{\bar{X}}$-algebra. It may be regarded as an $\mathcal{O}_{\bar{X}}$-subalgebra of $M_{p^n}(\mathcal{O}_{\bar{X}^\dagger})$ via Φ.*

(3) *The norm map N_A extends to a norm map*

$$N_A : A_{\bar{X}} \to \mathcal{O}_{\bar{X}^\dagger},$$

which is equal to $\det(\Phi(\bullet))$.

PROOF. (1) is a direct consequence of Lemma 2.2. (2) follows easily from (1). (3) is trivial. □

2.3. An ideal sheaf $\mathfrak{J}_{H^\dagger}$ of $A_{\bar{X}}$. In this subsection, we define an ideal $\mathfrak{J}_{H^\dagger}$ of $A_{\bar{X}}$ which is an analogue of the twisting sheaf of Serre.

DEFINITION 2.5. For any $c \in \mathbb{Z}$ we define a presheaf $\mathfrak{J}^c_{H^\dagger}$ on $\bar{X}$ by

$$\mathfrak{J}^c_{H^\dagger}(U) = \{s \in A_{\bar{X}}; \operatorname{ord}_{H^\dagger}(N_A(s)) \geq cp^n\}.$$

We can easily verify that the presheaf $\mathfrak{J}^c_{H^\dagger}$ above is actually a sheaf and that it is an $A_{\bar{X}}$*-bimodule.*

LEMMA 2.6. *Following statements are true.*

(1) *For any $c \in \mathbb{Z}$, $\mathfrak{J}^c_{H^\dagger}$ is a locally free left $A_{\bar{X}}$-module of rank one.*

(2) *For any $c_1, c_2 \in \mathbb{Z}$, we have*

$$\mathfrak{J}^{c_1}_{H^\dagger} \otimes_{A_{\bar{X}}} \mathfrak{J}^{c_2}_{H^\dagger} \cong \mathfrak{J}^{c_1+c_2}_{H^\dagger}.$$

Note that $\mathfrak{J}^{c_j}_{H^\dagger}$ $(j = 1, 2)$ are $A_{\bar{X}}$-bimodules so that we may take tensor products of them.

(3) *For any $c_1, c_2 \in \mathbb{Z}$ such that $c_1 < c_2$, $\mathfrak{J}^{c_1}_{H^\dagger}$ is a subsheaf of $\mathfrak{J}^{c_2}_{H^\dagger}$.*

PROOF. On each open set $\{T_j \neq 0\}$ of $\bar{X}$, $\mathfrak{J}_{H^\dagger}$ is generated by γ_j^{-c}.

□

There is another way of describing the ideal $\mathfrak{J}_{H^\dagger}$. We define

$$\rho_{H^\dagger} : A_{\bar{X}} \xrightarrow{\Phi} M_{p^n}(\mathcal{O}_{\bar{X}^\dagger}) \xrightarrow{\text{restr}} M_{p^n}(\mathcal{O}_{H^\dagger}).$$

Then with local calculations as in the proof of Lemma 2.2 and with the equation (AB) in the proof of Lemma 2.1, we see that the map $\rho_{H^\dagger}$ actually sends each section to a diagonal matrix. $\rho_{H^\dagger}$ thus gives an surjective map $\bar{\rho}_{H^\dagger} : A_{\bar{X}} \to \mathcal{O}_{H^\dagger}$ of sheaves.

LEMMA 2.7. *We have equalities*

$$\mathfrak{J}_{H^\dagger} = \mathrm{Ker}(\rho_{H^\dagger}) = \mathrm{Ker}(\bar{\rho}_{H^\dagger}).$$

Thus the sheaf $\mathfrak{J}_{H^\dagger}$ *is a both-sided ideal of* $A_{\bar{X}}$ *with* $A_{\bar{X}}/\mathfrak{J}_{H^\dagger} \cong \mathcal{O}_{H^\dagger}$.

PROOF. This is a direct consequence of the homomorphism theorem. □

PROPOSITION 2.8. *We have*

(1) $\mathfrak{J}_{H^\dagger}^k/\mathfrak{J}_{H^\dagger}^{k+1} \cong \mathcal{O}_{H^\dagger}(-k)$,
(2) $\mathfrak{J}_{H^\dagger}^p = A_{\bar{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-H)$.

PROOF. (1) On each open set $\{T_j \neq 0\}$ of $\bar{X}$, $\mathfrak{J}_{H^\dagger}$ is generated by γ_j^{-1}.
(2) Follows from the equation (I). □

2.4. Decomposition of $A_{\bar{X}}$ into line bundles. Let us define a subsheaf $\mathcal{L}_I$ of $A_{\bar{X}}$ by

$$\mathcal{L}_I(U) = \{f \in A_{\bar{X}}(U); f|_{X\cap U} \in \mathcal{O}_X(X \cap U) \cdot \gamma^I\}$$

DEFINITION 2.9. For any index set J, let us denote by $c_{p,J}$ the least integer which is not less than $|J|/p$. That means

$$c_{p,J} = \left\lceil \frac{|J|}{p} \right\rceil.$$

Then we easily have the following lemma.

LEMMA 2.10. *We have*

$$\mathcal{L}_I = \mathcal{O}(-c_{p,J}H)\gamma^J.$$

PROOF. Let U be an open set of $\bar{X}$. For any $a \in \mathcal{O}_X(U)$, we have

$$N_A(a\gamma^J) = N_A(a)N_A(\gamma^J) = a^{p^n}(t^J)^{p^n}.$$

Noting that $\mathrm{ord}_{H^\dagger} t_j = -1$ for any j, we see that the condition $\mathrm{ord}_{H^\dagger} N_A(a\gamma^J) \geq 0$ is equivalent to the inequality $\mathrm{ord}_{H^\dagger}(a) \geq |J|$ which is then equivalent to the inequality $\mathrm{ord}_H(a) \geq \lceil |J|/p \rceil = c_{p,J}$.

□

THEOREM 2.11. *The sheaf $A_{\bar{X}}$ decomposes into a direct sum of invertible sheaves as $\mathcal{O}_{\bar{X}}$-modules. More precisely, we have*

$$A_{\bar{X}} = \bigoplus_{I \in \mathbb{N}_p^{2n+m}} \mathcal{L}_I.$$

PROOF. Let s be a section of $A_{\bar{X}}$ on a non-empty open set U of $\bar{X}$. Then s may be written as a quotient

$$s = \frac{a}{z} \qquad \text{(for some } z \in Z_n, \text{ and } a \in A_n\text{)},$$

where a may be written as a sum

$$a = \sum_{I \in \mathbb{N}_p^{2n+m}} a_I \gamma^I \qquad (a_I \in Z_n).$$

Then we have

$$\deg(a) = \max_{I \in \mathbb{N}_p^{2n+m}} (\deg(a_I) + |I|).$$

Therefore, the condition $\mathrm{ord}_{H^\dagger}(N_A(s)) \geq 0$ holds if and only if the inequality $\deg(a_I) + |I| \geq \deg(z)$ holds for for any I. In other words, the condition $\mathrm{ord}_{H^\dagger}(N_A(s)) \geq 0$ is equivalent to the condition that $\mathrm{ord}_H(a_I\gamma^I/z) \geq 0$ holds for any I. The theorem follows easily from this equivalence. □

Essentially by repeating the argument above, we can generalize the above theorem as follows.

PROPOSITION 2.12. *For any $c \in \mathbb{Z}$, we have a direct sum decomposition*

$$\mathfrak{I}_{H^\dagger}^c = \bigoplus_{J \in \mathbb{N}_p^{2n+m}} \mathcal{O}(\lfloor \frac{c - |J|}{p} \rfloor)\gamma^J$$

as $\mathcal{O}_{\bar{X}}$-modules.

□

We note that the proposition is a non-commutative analogue of a result of Hartshorne [9, Corollary 6.4] which, roughly speaking, states that a direct image of a line bundle on a projective space by a Frobenius morphism is a direct sum of line bundles.

3. THE NORM-BASED EXTENSION SHEAF $W_{\bar{X}}$

3.1. Norms on W. In this subsection, we review some facts on norms on a reflexive modules of rank one over the algebra $A = A_{n,m}$. Proofs are mostly easy and details will be found in author's paper [16].

We first state the following proposition on reflexivity.

PROPOSITION 3.1. *Let W be an A-module of rank one. The following conditions are equivalent:*

(1) *W is A-reflexive. Namely, the "evaluation map" gives an isomorphism*

$$W \cong \mathrm{Hom}_{A\text{-rightmodule}}(\mathrm{Hom}_{A\text{-leftmodule}}(W, A), A).$$

(2) *W is A-torsion free and W_X is normal on X.*

(3) *W is A-torsion free and Z-reflexive.*

Proof. Almost identical with the proof of [16, Proposition 3.6] □

Definition 3.2. Let W be a reflexive A-module of rank one. A map $N : W \to S$ is a *norm* on W if the following conditions are satisfied.

(1) $N(ax) = N_A(a)N(x)$ for any $a \in A, w \in W$.

(2) For any Zariski open subset U of X, a section $s \in \Gamma(U, W_{\bar{X}})$ generates W_X if and only if $N_W(s)$ is not zero on $U^\dagger$.

Lemma 3.3. *A norm on W, if exists, is unique up to a constant multiple.*

Proof. Let N, N' be two norms on W. Let V be the open subset of X where W is locally free. Since W is reflexive and X is non-singular, $X \setminus V$ is of codimension at least 3 in X [10, Corollary 1.4].

By the property (2), we see that there exists an invertible function $f \in \Gamma(V^\dagger, \mathcal{O}_{X^\dagger}^\times)$ such that the equality $N(x) = N'(x)f$ holds. Since $X \setminus V$ is of codimension at least 3 in X, we see that f extends uniquely to an invertible function $f \in \Gamma(X^\dagger, \mathcal{O}_{X^\dagger}^\times) = \Bbbk^\times$. Note that $X^\dagger$ is isomorphic to an affine space $\mathbb{A}^{2n+m}$. □

Lemma 3.4. *Let J be a left ideal of A which is reflexive as an A-module. Then there exists an norm on J. Namely,*

$$N_J(\bullet) = N_A(\bullet)/c_J$$

where

$$c_J = \gcd(\{N_A(\chi); \chi \in J\}).$$

Proof. Almost identical with the proof of [16, Proposition 4.3]

□

Lemma 3.5. *Every reflexive left A-module W of rank one is isomorphic to an left ideal J of A.*

Proof. Since W is A-reflexive, we see that $\mathrm{Hom}_A(W, A)$ has a non-trivial section s. Then we may use s to embed W in A.

□

To sum up, we obtain the following proposition.

Proposition 3.6. *For any left A-module W of rank one, there exists a norm $N_W : W \to S$. It is unique up to a constant multiple.*

□

3.2. Definition of the norm-based extension $W_{\bar{X}}$. We now generalize Definition 2.3. Let W be a reflexive A-module of rank one.

DEFINITION 3.7. Let us define a norm-based extension sheaf $W_{\bar{X}}$ of W as

$$W_{\bar{X}}(U) = \{f \in W \otimes_Z K; f|_{U\cap X} \in W \otimes_Z \mathcal{O}(U \cap X), \quad N_W(f) \in \mathcal{O}(U)\}.$$

We shall see that the sheaf $W_{\bar{X}}$ is actually an $A_{\bar{X}}$-module.

LEMMA 3.8. *The sheaf $W_{\bar{X}}$ has the following properties.*

(1) *$W_{\bar{X}}$ is additive.*
(2) *$W_{\bar{X}}$ is a sheaf of $A_{\bar{X}}$-modules.*
(3) *$W_{\bar{X}}$ is reflexive as an $\mathcal{O}_{\bar{X}}$-module.*

PROOF. (1) Let $U \subset \bar{X}$ be an open set. If $W_{\bar{X}}(U) = 0$, then $W_{\bar{X}}(U)$ is surely additive. Otherwise, let $w_0 \in W_{\bar{X}}(U)$ be a non-zero section with the least valuation with respect to the hyperplane divisor H. Then

$$N_A(ww_0^{-1}) = N_W(w)N_W(w_0)^{-1}$$

is regular at general points of H. Thus for any $w_1, w_2 \in W_{\bar{X}}(U)$, there exists an open subset V of U such that the following conditions hold.

(i) $V \cap H \neq \emptyset$.
(ii) $N_W(w_1)N_W(w_0)^{-1} \in \mathcal{O}_{X^\dagger}(V)$.
(iii) $N_W(w_2)N_W(w_0)^{-1} \in \mathcal{O}_{X^\dagger}(V)$.

Thus $w_1w_0^{-1}$, $w_2w_0^{-1}$ are regular sections of $A_{\bar{X}}$ n V. By Lemma 2.2, we see that $(w_1 + w_2)w_0^{-1}$ is also a section of $A_{\bar{X}}(V)$. So in particular,

$$N_W(w_1 + w_2) = N_A\left((w_1 + w_2)w_0^{-1}\right) N_W(w_0) \in \mathcal{O}_{X^\dagger}(V).$$

On the other hand, w_1, w_2 are regular sections on $U \cap \mathbb{A}^{2n}$ so $N_W(w_1 + w_2)$ is regular on $U \setminus H$. That means, $N_W(w_1 + w_2)$ is regular except for a locus of codimension 2 in U. Thus $N_W(w_1 + w_2)$ is in $\mathcal{O}_{\bar{X}^\dagger}(U)$. This implies $w_1 + w_2 \in W_{\bar{X}}(U)$.
(2) It is easy to see that $W_{\bar{X}}(U)$ admits multiplications by elements of $A_{\bar{X}}(U)$. We thus conclude that $W_{\bar{X}}$ is a sheaf of $A_{\bar{X}}$-modules. □

3.3. Reflexive extensions of W. Let us suppose we are given a reflexive A-module W of rank one. Since W is a Z-module, we may consider a sheaf W_X on $X = \mathrm{Spec}(Z)$ associated to W. We have already shown that we have a reflexive extension $W_{\bar{X}}$ of the sheaf W_X to $\bar{X}$. In this subsection we prove that any other reflexive extension $\mathcal{F}$ of the sheaf W_X (to $\bar{X}$) is isomorphic to a Serre twist of $W_{\bar{X}}$. To do that, we first define a number $r_{\mathcal{F}}$ associated to $\mathcal{F}$.

Lemma 3.9. *Let $\mathcal{F}$ be a reflexive $A_{\bar{X}}$-module of rank one. We put $W = \mathcal{F}(X)$. For any open set U of $\bar{X}$ such that $U \cap H \neq \emptyset$ and $\mathcal{F}|_U$ is free, we put*

$$r_U = \min\{\operatorname{ord}_{H^\dagger} N_W(s); s \in \mathcal{F}(U)\}.$$

Then we have the following.

(1) *If s is a generating section of $\mathcal{F}$ on U as an A_X-module, then $\operatorname{ord}_{H^\dagger}(N_W(s)) = r_U$.*
(2) *$r_U = r_V$ if $U \subset V$ and hence r_U is independent of the choice of U.*

Proof. (1) is obvious since we have $(A_X|_U)s = \mathcal{F}|_U$. (2) We choose a generating section s of $\mathcal{F}$ on V and then compute r_V and r_U using (1). □

Definition 3.10. For a reflexive $A_{\bar{X}}$-module $\mathcal{F}$ of rank one, let us denote by $r_{\mathcal{F}}$ the number r_U in Lemma 3.9 (which is independent of the choice of U).

Lemma 3.11. *Let $\mathcal{F}$ be a reflexive $A_{\bar{X}}$-module of rank one. We put $W = \mathcal{F}(X)$. For any open set V of $\bar{X}$ such that $U \cap H \neq \emptyset$ and for any section $s \in \mathcal{F}(V \cap X)$, s is in* $\operatorname{Image}(\mathcal{F}(V) \overset{\text{restr}}{\to} \mathcal{F}(V \cap X))$ *if and only if* $\operatorname{ord}_{H^\dagger}(N_W(s)) \geq r_{\mathcal{F}}$.

Proof. We take an open subset U of V which satisfy the assumption of Lemma 3.9. Then the only if part is obvious from the definition of $r_{\mathcal{F}}$.

Assume the last inequality. Using the local arguments as above, we see that s is regular on an open subset U' of U where $\mathcal{F}$ is locally free. Since $\mathcal{F}$ is normal and the codimension of $U \setminus U'$ in U is at least 3 [10, Corollary 1.4], we see that s is regular on U. □

Proposition 3.12. *Let $\mathcal{F}$ be a reflexive $A_{\bar{X}}$-module of rank one. We put $W = \mathcal{F}(X)$. Then there exists an integer c such that an isomorphism*

$$\mathcal{F} \cong W_{\bar{X}} \otimes_{A_{\bar{X}}} \mathfrak{I}_{H^\dagger}^{c}$$

exists.

Proof. We may find an open subset U of $\bar{X}$ such that (i) $U \cap H \neq \emptyset$, (ii) $\mathcal{F}|_U$, $W_{\bar{X}}|_U$ is $A_{\bar{X}}$-free on U. Let us take local free generators $s_1 \in \mathcal{F}(U)$, $s_2 \in W_{\bar{X}}(U)$ of these sheaves. Then we see that there exists a rational section $x \in A \otimes_Z Q(Z)$ such that $xs_1 = s_2$ holds. Then by Lemma 2.1 we see that there exists an integer $c_{\mathcal{F}}$ such that

$$r_{\mathcal{F}} = r_{W_{\bar{X}}} + p^n c_{\mathcal{F}}$$

holds.

By extending the identity map on W we obtain an well-defined $A_{\bar{X}}$-linear morphism

$$\varphi : W_{\bar{X}} \otimes_A \mathfrak{I}_{H^\dagger}^{c_{\mathcal{F}}} \to \mathcal{F}$$

of sheaves. By definition, φ is an isomorphism on X and on the generic point of H. Thus φ is an isomorphism outside a locus of codimension at least 2 in $\bar{X}$. Since both $W_{\bar{X}} \otimes_A \mathfrak{I}_{H^\dagger}^{c_{\mathcal{F}}}$ and $\mathcal{F}$ are reflexive, and the base space $\bar{X}$ is normal, we conclude that φ is an isomorphism on the whole $\bar{X}$. □

3.4. Wrinkles of W.

PROPOSITION 3.13 ([10, Corollary 1.4]). *If the base space X is regular, then a reflexive sheaf on X is locally free except along a closed subset of codimension at least* 3.

LEMMA 3.14. *$W_{\bar X}/\mathfrak{J}_{H^\dagger}W_{\bar X}$ has no $A_{\bar X}/\mathfrak{J}_{H^\dagger}(\cong \mathcal{O}_{H^\dagger})$-torsion.*

PROOF. Let us assume that an section w of $W_{\bar X}$ satisfies

$$aw \in \mathfrak{J}_{H^\dagger}W_{\bar X}(U) \quad \text{for some } a \notin \mathfrak{J}_{H^\dagger}.$$

We may assume that U is small enough so that we may take a generating section $r \in \mathfrak{J}_{H^\dagger}(U)$ (which is typically one of the elements $\gamma_1^{-1}, \gamma_2^{-1}, \dots, \gamma_{2n+m}^{-1}$). Then we may write

$$aw = rw_1 \qquad \text{for some } w_1 \in W_{\bar X}(U).$$

This equation gives rise to the relation on norms

$$N_A(a)N_W(w) = N_A(r)N_W(w_1).$$

Thus $N_W(r^{-1}w) = N_A(a)^{-1}N_W(W_1)$ is regular at the generic point of H. On the other hand, since r is a generating section of $\mathfrak{J}_{H^\dagger}$, we see that $r^{-1}w$ is regular on $U \cap X$. By a "codimension at least 2 argument", we see that $r^{-1}w$ is regular on the whole U. Thus we see that $w = r \cdot (r^{-1}w)$ is in $\mathfrak{J}_{H^\dagger}W_{\bar X}(U)$, as required.

□

The double dual L_W of $W_{\bar X}/\mathfrak{J}_{H^\dagger}W_{\bar X}$ as an $\mathcal{O}_{H^\dagger}$-module is a reflexive $\mathcal{O}_{H^\dagger}$-module of rank one and thus isomorphic to a line bundle of the form $\mathcal{O}_{H^\dagger}(c_W)$ for some integer c_W. Namely, we have an inclusion

$$W_{\bar X}/\mathfrak{J}_{H^\dagger}(W_{\bar X}) \hookrightarrow L_W \cong \mathcal{O}_{H^\dagger}(c_W).$$

We may then have a unique closed subscheme F of $H^\dagger$ such that its ideal sheaf I_F satisfies the relation

$$W_{\bar X}/\mathfrak{J}_{H^\dagger}(W_{\bar X}) \cong I_F\mathcal{O}_{H^\dagger}(c_W).$$

DEFINITION 3.15. Let us call F the *wrinkle* of W and denote it as Wrinkle(W). The number c_W is called the *degree of norm-based extension* of W.

DEFINITION 3.16. For any $A_{\bar X}$-module $\mathcal{F}$, we use the following notation.

$$\mathrm{Sing}_{A_{\bar X}}(\mathcal{F}) = \{p \in \bar X; \mathcal{F} \text{ is not } A_{\bar X}\text{-locally free near } p\}.$$

$$\mathrm{Sing}_{\mathcal{O}_{\bar X}}(\mathcal{F}) = \{p \in \bar X; \mathcal{F} \text{ is not } \mathcal{O}_{\bar X}\text{-locally free near } p\}.$$

We may introduce scheme structures on these spaces, but we do not go further and content ourselves by noting that they are closed subset of $\bar{X}$. It follows easily from Theorem 2.11 that the inclusion

$$\operatorname{Sing}_{A_{\bar{X}}}(\mathcal{F}) \subset \operatorname{Sing}_{\mathcal{O}_{\bar{X}}}(\mathcal{F})$$

holds.

In what follows, for any scheme S, we denote by $|S|$ the underlying space of S.

LEMMA 3.17. *Let W be a reflexive A-module of rank one. Then we have the following.*

(1) $\operatorname{Wrinkle}(W)$ *is a closed subset of codimension at least* 2 *in* $H^\dagger$.
(2) $|\operatorname{Wrinkle}(W)| = |\operatorname{Sing}_{A_{\bar{X}}}(W)| \cap H$.
(3) *More generally, for any reflexive extension $\mathcal{F}$ of W_X, we have* $|\operatorname{Wrinkle}(W)| = |\operatorname{Sing}_{A_{\bar{X}}}(\mathcal{F})| \cap H$.

PROOF. (1) follows immediately from the definition of the wrinkle.

(2) Let us take a point $p \in H^\dagger \setminus \operatorname{Wrinkle}(W)$. By the definition of the wrinkle, we see that there exists an open neighborhood U of p in $\bar{X}$ and a section s of $W_{\bar{X}}$ defined on U such that s generates $(W_{\bar{X}}/\mathfrak{I}_{H^\dagger} W_{\bar{X}})|_U$. It is then easy to see by using Nakayama's lemma that s generates $W_{\bar{X}}$ as an $A_{\bar{X}}$-module on a neighborhood of p in $\bar{X}$. It is even easier to prove the converse inclusion. (3) follows from (2) and Proposition 3.12

□

3.5. An example. As an illustration of computations of wrinkles, we give an example. Let us consider the case where $n = 1, m = 1$ and put $A = A_{1,1}$. We denote the standard generators of A by ξ, η, v instead of γ_1, γ_2, t_3. The eigenvalue of ξ (resp. η) is denoted by t (resp. u) instead of t_1 (resp. t_2).

For a positive integer $l < p$, let us put

$$J_l^{(1,1)} = A(\xi\eta - l) + A\eta^{l+1}$$

and compute its wrinkle. The ideal sheaf $J_l^{(1,1)}$ is obviously obtained by an adjunction of a variable v to an ideal

$$J_l^{(1,0)} = A_{1,0}(\xi\eta - 1) + A_{1,0}\eta^{l+1}$$

of an ordinary non-degenerate Weyl algebra $A_{1,0}$ in two variables, which is also known as the "first Weyl algebra". Since any reflexive sheaf on a smooth scheme of dimension at least 2 is locally free [10], the wrinkle of $J_l^{(1,0)}$ is void. What follows in this subsection may be viewed as an way to obtain an invariant of such objects. One may also use ordinary Weyl algebra $A_{2,0} \supset A_{1,0}$ in 4-variables, that is, "the second Weyl algebra, instead of the somewhat unfamiliar degenerate Weyl algebra to obtain an invariant in a similar manner.

LEMMA 3.18. $J_l^{(1,1)} \cong A\xi^l \cap A\eta$ *as an A-module. More precisely, we have*

$$J_l^{(1,1)}\xi^l = A\xi^l \cap A\eta.$$

Proof. This is a well-known result. See for example [11]. For the sake of completeness, we give here a proof as we shall use a similar method later in Lemma 3.22. We first note that $(\xi\eta - l)\xi^l = \xi^l\eta$ is in $A\eta$. It is also easy to see that $\eta^{l+1}\xi^l \in A\eta$. So we surely have $J_l^{(1,1)}\xi^l \subset A\xi^l \cap A\eta$.

To see the converse inclusion, we notice firstly that

$$\xi^{a+1}\eta^{b+1} = (\xi^a\eta^b)(\xi\eta - l) + (\text{terms of degree } \leq a+b)$$

holds so that we see by an induction that any element $\chi \in A$ may be written as

$$\chi = c(\xi,\eta,v)\cdot(\xi\eta - l) + a(v,\xi)\xi + b(v,\eta)$$

for some $c(\xi,\eta,v) \in A$, $a(v,\xi) \in \Bbbk\langle v,\xi\rangle$, $b(v,\eta) \in \Bbbk\langle v,\eta\rangle$. By expanding $b(v,\eta)$ in terms of η, we may further write

$$\chi = c(\xi,\eta,v)\cdot(\xi\eta - l) + b_1(v,\eta)\eta^{l+1}a(v,\xi)\xi + b_0(v,\eta)$$

for some $b_0(v,\eta) \in \sum_{j=0}^{l}\Bbbk[v]\eta^j$ and $b_1(v,\eta) \in \Bbbk\langle v,\eta\rangle$. Now assume $\chi\xi^l \in A\eta$. Then we see immediately that an element $a(v,\xi)\xi^{l+1} + b_0(v,\eta)\xi^l$ is in $A\eta$. By writing down the element as a $\Bbbk[v]$-linear combination of $\{\xi^s\eta^t\}_{s,t\in\mathbb{N}}$, we see that a and b_0 should be zero.

□

Lemma 3.19. *The norm $N_{J_l}^{(1,1)}$ of the module $J_l^{(1,1)}$ is given by*

$$N_{J_l^{(1,1)}}(\bullet) = N_A(\bullet)/N_A(\eta).$$

Proof. Let us put $\tilde{J} = J_l^{(1,1)}\xi^l$. By the lemma above, we see that $N_A(\xi^l)|N_A(\chi)$ and $N_A(\eta)|N_A(\chi)$ hold for any element $\chi \in \tilde{W}$. Thus $t^{pl}u^p|N_A(\chi)$ for all $\chi \in \tilde{W} = W\xi^l$. Thus $u^p|N_A(\chi)$ for all $\chi \in \tilde{W}$. Since $J_l^{(1,1)}$ has elements $\xi\eta - l, \eta^{l+1}$ whose norms by N_A are $t^pu^p, u^{p(l+1)}$, respectively, we see that $c_W = u^p$.

□

Lemma 3.20. *We have the following.*

(1) $N_{J_l^{(1,1)}}(\xi\eta - l) = t^p$.
(2) $N_{J_l^{(1,1)}}(\eta^{l+1}) = u^{pl}$.
(3) $N_{J_l^{(1,1)}}(\xi\eta - l + c\eta^p) = t^p + c^pu^{p^2-p} - c$ *for any $c \in \Bbbk$.*

Proof. (2) is immediate.

$$(\xi\eta - l + c\eta^p)^p - (\xi\eta - l + c\eta^p) = \xi^p\eta^p + c^p\eta^{p^2} - c\eta^p = t^pu^p + c^pu^{p^2} - cu^p$$

gives the minimal polynomial of $(\xi\eta - 1 + c\eta^p)$. We may thus see that

$$N_A(\xi\eta - l + c\eta^p) = t^pu^p + c^pu^{p^2} - cu^p$$

holds. This proves (3). (1) is a special case of (3). □

Corollary 3.21. *$J_l^{(1,1)}$ is A_X-locally free on X.*

PROOF. Norms $N_{J_l^{(1,1)}}(\xi\eta - l)$, $N_{J_l^{(1,1)}}(\eta^{l+1})$, $N_{J_l^{(1,1)}}(\xi\eta - l + \eta^p)$ have no common zero locus on X. Thus each of the elements $\xi\eta - l, \eta^{l+1}, \xi\eta - l + \eta^p$ generates W_X on an open set where its norm is non-zero. □

To deal with the wrinkle of $J_l^{(1,1)}$, we need to consider a projective space $\bar{X}$ which completes X. Namely, we consider the homogeneous coordinate $(T : U : V : T_0)$ such that $T/T_0 = t, \quad U/T_0 = u, \quad V/T_0 = v$ hold. $\bar{X}$ is then equal to $\mathrm{Proj}(\Bbbk[T, U, V, T_0])$. In view of Corollary 3.21, we see

$$|\operatorname{Sing}(W_{\bar{X}})| \subset |H| = \{T_0 = 0\}.$$

Furthermore, by viewing norms, we may easily verify that an element $\xi^{-1}(\xi\eta - l)$ is an $A_{\bar{X}}$-base of $W_{\bar{X}}$ at $\{T \neq 0\}$ and that an element $\eta^{-l}\eta^{l+1}$ is an $A_{\bar{X}}$-base of $W_{\bar{X}}$ at $\{U \neq 0\}$.

So the only place we need to investigate the behavior of $(J_l^{(1,1)})_{\bar{X}}$ is a neighborhood of $(T : U : V : T_0) = (0 : 0 : 1 : 0)$. So we concentrate ourselves on an open set $\tilde{X} = \{V \neq 0\}$. Let us put

$$x = T/V, y = U/V, z = T_0/V,$$

and let us put further

$$\tilde{\xi} = v^{-1}\xi, \quad \tilde{\eta} = v^{-1}\eta, z = v^{-1}.$$

Then the following lemma holds.

LEMMA 3.22. *Let us consider an affine open set $\tilde{X} = \{V \neq 0\}$ of $\bar{X}$. Then we have the following.*

1. *(x, y, z) is a local coordinate of $\tilde{X}^\dagger$.*
2. *$\tilde{\xi}, \tilde{\eta}, z$ generate $\tilde{A} = \Gamma(\tilde{X}, A_{\bar{X}})$.*
3. *z is in the center of $\tilde{A}$. $[\tilde{\eta}, \tilde{\xi}] = z^2$.*
4. *$\tilde{J} = \Gamma(\tilde{X}, (J_l^{(1,1)})_{\bar{X}})$ is isomorphic to $\tilde{A}\tilde{\xi}^l \cap \tilde{A}\tilde{\eta}$ and is also isomorphic to $\tilde{A}(\tilde{\xi}\tilde{\eta} - z^2 l) + \tilde{A}\tilde{\eta}^{l+1}$.*

PROOF. Statements (1), (2), (3) follow easily from direct computations. The statement (4) is shown by a same method as was used in the proof of Lemma 3.18

□

PROPOSITION 3.23. *The wrinkle* $\mathrm{Wrinkle}(J_l^{(1,1)})$ *is equal to* $\mathrm{Spec}(\Bbbk[x, y]/(x, y^l))$. *In particular we see that the wrinkle is an invariant good enough to distinguish $J_l^{(1,1)}$ for $l \in \{1, 2, 3, \ldots, p-1\}$.*

PROOF. The inclusion map

$$J_l^{(1,1)} \cong \left(\tilde{A}(\tilde{\xi}\tilde{\eta} - z^2 l) + \tilde{A}\tilde{\eta}^{l+1}\right) \hookrightarrow \tilde{A}$$

induces a map

$$f : J_l^{(1,1)}/zJ_l^{(1,1)} \to L = \tilde{\eta} \cdot (A_{\bar{X}}/zA_{\bar{X}})$$

of $A_{\bar{X}}/zA_{\bar{X}}(\cong \mathcal{O}_{H^\dagger})$-modules defined over $\tilde{X}^\dagger$. We see immediately that f is well defined and may be regarded as a morphism of $\mathcal{O}_{H^\dagger}$-modules defined on $H^\dagger \cap \tilde{X}^\dagger$. It is easy to verify the following facts.

(1) The sheaf L is invertible on $H^\dagger \cap \tilde{X}^\dagger$.
(2) f is injective.
(3) f is surjective except for a locus of codimension at least 2.

Thus we see that the double dual of $J_l^{(1,1)}/zJ_l^{(1,1)}$ is equal to L. The proposition follows easily from this. □

4. MAIN THEOREM

If W is a trivial A-module, then its norm-based extension $W_{\bar{X}}$ is a trivial $\mathcal{O}_{\bar{X}}$-module so that the wrinkle $\mathrm{Wrinkle}(W_{\bar{X}})$ is empty. When $\dim(X) \geq 3$, the converse is also true as the following theorem states.

THEOREM 4.1. *Assume* $\dim(X) \geq 3$. *Let W be a reflexive A-module of rank one. If W has no wrinkle, that means* $\mathrm{Wrinkle}(W) = \emptyset$, *then W is trivial.*

PROOF. Assume $\mathrm{Wrinkle}(W) = \emptyset$. Then by using Lemma 3.17 we see that the sheaf $W_{\bar{X}}$ is locally free on an open neighborhood of H. So $W_{\bar{X}}/\mathfrak{J}_{H^\dagger}W_{\bar{X}}$ is locally free over $A_{\bar{X}}/\mathfrak{J}_{H^\dagger} \cong \mathcal{O}_{H^\dagger}$. In other words,

$$W_{\bar{X}}/\mathfrak{J}_{H^\dagger}W_{\bar{X}} \cong \mathcal{O}_{H^\dagger}(c) \text{ for some } c.$$

Let us put $\mathcal{F} = \mathfrak{J}^c_{H^\dagger} \otimes_A W_{\bar{X}}$ so that we have $\mathcal{F}/\mathfrak{J}_{H^\dagger}\mathcal{F} \cong \mathcal{O}_{H^\dagger}$. Let us consider a filtration of $\mathcal{F}$:

$$\text{(FF)} \qquad \mathcal{F}_{\bar{X}} \supset \mathfrak{J}_{H^\dagger}\mathcal{F}_{\bar{X}} \supset \mathfrak{J}^2_{H^\dagger}\mathcal{F}_{\bar{X}} \supset \mathfrak{J}^3_{H^\dagger}\mathcal{F}_{\bar{X}} \supset \cdots \supset \mathfrak{J}^p_{H^\dagger}\mathcal{F}_{\bar{X}}.$$

The associated graded module looks like

$$\mathfrak{J}^k_{H^\dagger}\mathcal{F}/\mathfrak{J}^{k+1}_{H^\dagger}\mathcal{F} \cong (\mathfrak{J}^k_{H^\dagger}/\mathfrak{J}^{k+1}_{H^\dagger}) \otimes_{A_{\bar{X}}} \mathcal{F} \cong (\mathfrak{J}^k_{H^\dagger}/\mathfrak{J}^{k+1}_{H^\dagger}) \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{F}/\mathfrak{J}_{H^\dagger}\mathcal{F} \cong \mathcal{O}_{H^\dagger}(-k).$$

We note that each module $\mathcal{O}_{H^\dagger}(-k)$ is, being a line bundle on $H^\dagger$, *a direct sum of line bundles on H* [9, Corollary 6.4]. Since $H^1(\mathbb{P}^n; \mathcal{O}(\bullet)) = 0$ for any line bundle $\mathcal{O}(\bullet)$, all of the extension groups involved in the filtration (FF) vanish. Thus we have

$$i_H^*(\mathcal{F}) \cong \bigoplus_{k=0}^{p-1} \mathcal{O}_{H^\dagger}(-k)$$

as *bundles over H*. That means, we regard each line bundle $\mathcal{O}_{\bar{X}^\dagger}(-kH^\dagger)$ on $H^\dagger$ which appears at the right-hand side as a direct sum of line bundles on H.

In particular, $i_H^*(\mathcal{F})$ is a direct sum of line bundles on H. This implies, via the theorem of Abe-Yoshinaga "Reflexive Horrocks" [1], that the sheaf $\mathcal{F}$ is a direct sum of line bundles on the whole $\bar{X}$. In other words, we have an isomorphism

$$\mathcal{F} \cong \bigoplus_{k=0}^{p-1} \mathcal{O}_{\bar{X}^\dagger}((-k)H^\dagger)$$

of $\mathcal{O}_{\bar{X}}$-modules. In particular, there is a non-vanishing global section $m_0 \in \Gamma(\bar{X}, \mathcal{F})$ which is unique up to a constant multiple. m_0 then restricts to a non-zero section of $\mathcal{F}/\mathfrak{J}_{H^\dagger}\mathcal{F} \cong \mathcal{O}_{H^\dagger}$. Thus m_0 is a generator of $\mathcal{F}_{\bar{X}}/\mathfrak{J}_{H^\dagger}\mathcal{F}_{\bar{X}}$. This means that m_0 is a generator of $\mathcal{F}_{\bar{X}}$ on a neighborhood of H. So $N_W(m_0)$ is not zero everywhere on X, since, otherwise, the zero locus of $N_W(m_0)$ intersects H nontrivially. We conclude that $\mathcal{F}_{\bar{X}}$ is generated by m_0 over $A_{\bar{X}}$. That means, $\mathcal{F}_{\bar{X}}$ is a trivial $A_{\bar{X}}$-module. □

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