ON d-VERY AMPLE LINE BUNDLES ON ABELIAN SURFACES

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ABSTRACT. Let (X, L) be a polarized abelian surface over an algebraically closed field k. We give a geometric characterization of (X, L) such that (1) L is not spanned with $L^2 \geq 6$, (2) X is defined over k with $\operatorname{char} k \neq 2$, and L is spanned but not very ample with $L^2 \geq 10$, (3) X is defined over $\mathbb{C}$, and L is very ample but not 2-very ample with $L^2 \geq 14$, and (4) X is defined over $\mathbb{C}$, and L is 2-very ample but not 3-very ample with $L^2 \geq 18$.

§ 0. Introduction.

Let (X, L) be a polarized manifold, that is, a pair consisting of a smooth projective variety X and an ample line bundle L on X defined over an algebraically closed field k. One of the most important problems concerning polarized manifolds is to study properties of their adjoint bundles $K_X + mL$, such as freeness, very ampleness, or more generally, d-very ampleness introduced by Beltrametti and Sommese ([12, 8.5], see also Definition 1.1). The last one is a natural generalization of freeness and very ampleness. It is known that freeness (resp. very ampleness) is equivalent to 0-very ampleness (resp. 1-very ampleness).

In the case where X is a complex surface, this problem has been studied by Reider and Beltrametti and Sommese and now is known a fairly satisfactory numerical criterion for $K_X + L$ to be d-very ample under the natural assumption that $L^2 \geq 4d + 6$ ([27], [11]). This result, which is called the Reider type Theorem, has been also generalized to a surface in positive characteristic to a large extent ([28], [24], see also Theorem 1.3 for a special case).

However, from a somewhat different point of view, it is also interesting to study when the adjoint line bundle is $(d-1)$-very ample but not d-very ample especially in the case where d is small. This problem shares a similar flavour with a classification problem of polarized manifolds whose adjoint bundle $K_X + mL$ are not nef for large $m \leq \dim X + 1$.

The purpose of this paper is to study the problem especially for polarized abelian surfaces (X, L) with $L^2 \geq 4d+6$ and to give explicit geometric descriptions of them in the following four cases:

(1) $d = 0$ i.e. L is not free and $\operatorname{char} k$ is arbitrary (Theorem 2.1);
(2) $d = 1$ i.e. L is free but not very ample and $\operatorname{char} k \neq 2$ (Theorem 2.2);
(3) $d = 2$ and $\operatorname{char} k = 0$ (Theorem 2.6);
(4) $d = 3$ and $\operatorname{char} k = 0$ (Theorem 2.7).

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Theorem 2.1 is a generalization of the well known result ([21]) over $\mathbb{C}$ to an arbitrary base field. Theorem 2.2 will give not only a generalization but also a refinement of the result of Hulek-Lange ([19]) over $\mathbb{C}$. Theorems 2.6 and 2.7 are related to the works of Bauer-Szemberg ([4], [5]) and Terakawa ([29]) concerning d-very ampleness of a polarized complex abelian surface.

A brief outline of this paper is as follows.

First we formulate the Reider type Theorem (Theorem 1.3) for a polarized abelian surface in arbitrary characteristic. For Theorem 2.1 and 2.2, we apply this to find an elliptic curve D and then make a fibration $X \to X/D$. By examining this fibration, we obtain the results. Since Theorem 1.3 is characteristic free, our argument should be also characteristic free for the most part. However, our proof of Theorem 2.2 involves a double covering method in some part. This is the reason why we assume $\operatorname{char} k \neq 2$ in Theorem 2.2. In order to prove Theorems 2.6 and 2.7, we make use of Theorem 1.5 due to Terakawa ([29]) instead of Theorem 1.3. Theorem 1.5 is much sharper than Theorem 1.3, while it is proved only in characteristic 0. Even in positive characteristic, we may apply Theorem 1.3 to restrict possible candidates (X, L) for $d = 2, 3$. However in these cases, it is hard to find out for which pairs L are really $(d-1)$-very ample but not d-very ample. For this reason, we assume $\operatorname{char} k = 0$ in Theorems 2.6 and 2.7.

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§ 1. Preliminaries.

Definition 1.1. Let (X, L) be a polarized surface defined over k. Then L is called d-very ample if for any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ with $\operatorname{length} \mathcal{O}_Z = d + 1$, the map

$$\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_Z)$$

is surjective.

Lemma 1.2. *Let X be an abelian surface over k, and L an ample line bundle on X. Then $h^1(-L) = 0$.*

Proof. If $\operatorname{char} k = 0$, then this is Kodaira's Vanishing Theorem. If $\operatorname{char} k > 0$, then see e.g. Corollary 8 in [28].

Theorem 1.3. (Reider, Beltrametti-Sommese, Shepherd-Barron, Nakashima) *Let (X, L) be a polarized abelian surface defined over k with $L^2 \geq 4d + 6$. Then L is d-very ample unless there exists an effective divisor D on X such that*

$$LD - d - 1 \leq D^2 < \frac{LD}{2} < d + 1.$$

Proof. If $\operatorname{char} k = 0$, see [10]. If $\operatorname{char} k > 0$, then by using Theorem 7 (i) in [28], we can prove this Theorem by the same method as in the proof of Theorem 3.1 in [24]. $\square$

Proposition 1.4. *Let (X, L) be a polarized surface over k. Assume that* (1) *L is d-very ample and* (2) *$k = \mathbb{C}$ if $d \geq 2$. Then $LC \geq d$ for any irreducible curve C on X. If $LC \leq d+1$ for an irreducible reduced curve, then $C \cong \mathbb{P}^1$.*

Proof. See Corollary (1.3) and Proposition (1.4) in [11].

Theorem 1.5. (Terakawa) *Let (X, L) be a polarized abelian surface over $\mathbb{C}$. Then L is d-very ample if and only if $L^2 \geq 4d+6$ and there exists no effective divisor D satisfying the inequalities*

$$LD \leq 2g(D) + d - 1 \leq 2d + 1.$$

Proof. Here we give the Terakawa's proof for convenience. (See also Theorem 3.15 in [29].) By Theorem 1.3, we can easily prove the "if" part. So we prove the "only if" part. Since X is an abelian surface, we obtain $L^2 = 2h^0(L)$. By Lemma 2.8 in [3] we have $h^0(L) \geq 2d+3$. Hence $L^2 \geq 4d+6$. Assume that there exists an effective divisor D which satisfies the above inequalities. By Proposition 1.4, D is irreducible and reduced. If $d \leq 1$, then $g(D) = 1$ and D is a smooth elliptic curve since X is an abelian surface. Furthermore we have $LD \leq d+1$. But by Proposition 1.4, this is a contradiction. If $d \geq 2$, then it follows from Theorem 1.7 in [11] that $g(D) \geq d+2$ since $LD \leq 2g(D) + d - 1$. Then we have $3d+3 \leq 2g(D) + d - 1 \leq 2d+1$. This is impossible. $\square$

Remark 1.5.1. In [5], Bauer and Szemberg gave a precise criterion for an ample line bundle of type $(1, t)$ on an abelian surface with Picard number 1 to be d-very ample.

Lemma 1.6. *Let X be an abelian surface defined over k and let L be an ample line bundle on X. Assume that* (1) *L is d-very ample and* (2) *$k = \mathbb{C}$ if $d \geq 2$. If D is an effective divisor on X such that $D^2 = 0$ and $LD \leq d+2$, then D is a smooth elliptic curve.*

Proof. Assume that D is not irreducible. Then there exists an irreducible component C of D such that $LC \leq d+1$. By Proposition 1.4 we get $D \cong \mathbb{P}^1$. But since X is an abelian surface, this is impossible. By the same argument we can prove that D is reduced. Since $D^2 = 0$, we obtain that $g(D) = 1$. If D has a singular point, then D is rational. But this is impossible because X is an abelian surface. Therefore D is smooth. $\square$

Lemma 1.7. *Let X be an abelian surface defined over k. Assume that X contains a smooth elliptic curve D. Then there exists an elliptic fibration $f : X \to C$ such that C is a smooth elliptic curve, any fiber of f is smooth, and D is a fiber of f.*

Proof. By a translation of D, we may assume that D contains the origin of X and D is an abelian subvariety of X. Then there exist the quotient X/D and the surjective homomorphism $f : X \to X/D$. Then X/D is a smooth elliptic curve and every fiber of f is isomorphic to the fiber over the origin of X/D which is D by construction. $\square$

Lemma 1.8. *Let X be an abelian surface over k. Assume that there exist a smooth elliptic curve C and a surjective morphism $f : X \to C$ with connected fibers such that f has a section S. Then $X \cong C \times F$, f is identified with the first projection via this isomorphism, and S is a fiber of the second projection, where F is a fiber of f.*

Proof. We remark that f is an elliptic fibration such that any fiber of f is smooth. Let S be a section of f. Then by Lemma 1.7, there exist a smooth elliptic curve C' and an elliptic fibration $h : X \to C'$ such that any fiber of h is a smooth elliptic curve and S is a fiber of h. Moreover any fiber of h (resp. f) is a section of f (resp. h). In particular $C' \cong F$. Then there exists a morphism $\pi : X \to C \times C'$ such that $f = p_1 \circ \pi$ and $h = p_2 \circ \pi$, where p_1 (resp. p_2) is the projection $C \times C' \to C$ (resp. $C \times C' \to C'$). We remark that π is bijective by construction. Let $F_f = f^*(x)$ and $F_h = h^*(y)$, where $x \in C$ and $y \in C'$. Then $F_f = \pi^* \circ p_1^*(x)$ and $F_h = \pi^* \circ p_2^*(y)$. Then

$$\begin{aligned} 1 = F_f F_h &= (\pi^* \circ p_1^*(x))(\pi^* \circ p_2^*(y)) \\ &= \deg(\pi)(p_1^*(x)p_2^*(y)). \end{aligned}$$

Hence π is birational. Therefore by Zariski Main Theorem, we obtain that π is an isomorphism. □

Lemma 1.9. *Let X and Y be smooth projective surfaces over k and let $\pi : X \to Y$ be a double covering. Assume that* $\operatorname{char} k \neq 2$. *Then the following hold.*

1. *The branch locus of π is smooth and linearly equivalent to $2\mathcal{N}$ for some $\mathcal{N} \in \operatorname{Pic}(Y)$.*
2. *$K_X = \pi^*(K_Y + \mathcal{N})$.*
3. *$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{N}^{-1}$.*
4. *$h^i(\pi^*\mathcal{L}) = h^i(\mathcal{L}) + h^i(\mathcal{L} \otimes \mathcal{N}^{-1})$ for any nonnegative integer i and $\mathcal{L} \in \operatorname{Pic}(Y)$.*
5. *$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(\mathcal{N}^{-1})$.*

Proof. See Chap. V, § 22 in [2], (6.11) in [17], or [25]. □

Proposition 1.10. *Let (X, L) be a polarized abelian surface defined over k. If $g(L) = 2$, then (X, L) is one of the following:*

1. *$X \cong J(B)$, and L is a translation of B, where B is a smooth projective curve of genus two, and $J(B)$ is the jacobian variety of B.*
2. *$X \cong E_1 \times E_2$, and $L = p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2$, where E_1 and E_2 are smooth elliptic curves, $p_i : E_1 \times E_2 \to E_i$ is the i-th projection, and $\mathcal{D}_i \in \operatorname{Pic}(E_i)$ with $\deg \mathcal{D}_1 = \deg \mathcal{D}_2 = 1$.*

Proof. This is a corollary of Matsusaka-Ran's criterion (see [15]). □

Lemma 1.11. *Let B be a smooth projective curve of genus two and $J(B)$ is the jacobian variety. If a divisor A is numerically equivalent to B, then $AN \geq 2$ for any irreducible curve N.*

Proof. By hypothesis, we may assume that $A = B$. If $BN < 2$ for some irreducible curve N, then $BN = 1$ since B is ample. By Hodge index Theorem, we obtain

$N^2 = 0$ and N is a smooth elliptic curve because X is an abelian surface. By Lemma 1.7, there exists an elliptic fibration $f : X \to C$ such that any fiber of f is smooth and N is a fiber of f. But since $BF = BN = 1$, B is a section of f and this is impossible because C is a smooth elliptic curve. $\square$

§ 2. Main Results.

Theorem 2.1. *Let (X, L) be a polarized surface defined over k. Then X is an abelian surface and L is not spanned with $L^2 \geq 6$ if and only if $X \cong E_1 \times E_2$ and $L = p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2$ with $\deg \mathcal{D}_1 = 1$ and $\deg \mathcal{D}_2 \geq 3$, where E_1 and E_2 are smooth elliptic curves, $p_i : E_1 \times E_2 \to E_i$ is the i-th projection, and $\mathcal{D}_i \in \operatorname{Pic}(E_i)$ for $i = 1, 2$.*

Proof. We can easily prove the "if" part. So we prove the "only if" part. By Theorem 1.3, there exists an effective divisor D on X such that $LD = 1$ and $D^2 = 0$. In particular, D is irreducible and reduced. Since X is an abelian surface and $g(D) = 1$, D is a smooth elliptic curve. By Lemma 1.7, there exist a smooth elliptic curve C and an elliptic fibration $f : X \to C$ such that any fiber of f is smooth and D is a fiber of f. On the other hand, by Lemma 1.2 and Riemann-Roch Theorem, we have $h^0(L) = L^2/2 \geq 3$. Hence f has a section S since $LF = LD = 1$. We remark that $L = S + f^*\mathcal{B}$ for some $\mathcal{B} \in \operatorname{Pic}(C)$ with $\deg \mathcal{B} \geq 3$. By Lemma 1.8, we get $X \cong C \times F$, f is the first projection via this isomorphism, and S is a fiber of the second projection $h : X \to F$, where F is a fiber of f.

This completes the proof of Theorem 2.1. $\square$

Remark 2.1.1. If $k = \mathbb{C}$, then Theorem 2.1 is well-known. (See e.g. [21].)

Theorem 2.2. *Let (X, L) be a polarized surface over k with $\operatorname{char} k \neq 2$. Then X is an abelian surface and L is spanned but not very ample with $L^2 \geq 10$ if and only if (X, L) satisfies one of the following:*
(1) $X \cong E_1 \times E_2$ and $L = p_1^*\mathcal{B}_1 \otimes p_2^*\mathcal{B}_2$ with $\deg \mathcal{B}_1 \geq 3$ and $\deg \mathcal{B}_2 = 2$.
(2) There exists a surjective morphism $f : X \to C$ with connected fibers such that C is a smooth elliptic curve and any fiber of f is a smooth elliptic curve, and (X, L) is one of the following:

- (2-1) *There exists an ample spanned vector bundle $\mathcal{E}$ of rank two on C such that $\mathcal{E} = \mathcal{E}' \otimes \mathcal{M}_1$ with $\deg \mathcal{M}_1 \geq 3$, X is a double covering of $\mathbb{P}(\mathcal{E})$ whose branch locus is smooth and linearly equivalent to $-2K_{\mathbb{P}(\mathcal{E})}$, $f = p \circ \pi$, and $L = \mathcal{O}(T) \otimes f^*\mathcal{M}_1$,*
- (2-2) *$X \cong J(B)$ and $L = \mathcal{O}_X(A) \otimes f^*\mathcal{M}_2$ such that A is a translation of B, $AF = 2$ for a fiber F of f, and $\deg \mathcal{M}_2 \geq 2$,*
- (2-3) *$X \cong E_1 \times E_2$ and $L = p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2 \otimes f^*\mathcal{M}_3$ such that any fiber of p_i is a section of f for $i = 1, 2$, $\deg \mathcal{D}_1 = \deg \mathcal{D}_2 = 1$, and $\deg \mathcal{M}_3 \geq 2$,*

where in (1) and (2-3) E_1 and E_2 are smooth elliptic curves, $p_i : E_1 \times E_2 \to E_i$ is the i-th projection, $\mathcal{B}_i, \mathcal{D}_i \in \operatorname{Pic}(E_i)$ for $i = 1, 2$, $\mathcal{M}_3 \in \operatorname{Pic}(C)$, in (2-1) $\mathcal{E}' = \mathcal{O}_C \oplus \mathcal{L}_1$ for $\mathcal{L}_1 \in \operatorname{Pic}(C)$ with $\mathcal{L}_1 \not\cong \mathcal{O}_C$ and $2\mathcal{L}_1 \cong \mathcal{O}_C$, $\mathcal{M}_1 \in \operatorname{Pic}(C)$, $p : \mathbb{P}(\mathcal{E}) \to C$ is the natural projection, $\pi : X \to \mathbb{P}(\mathcal{E})$ is the double covering, and T is a smooth elliptic curve with $TF = 2$ for a fiber F of f, and in (2-2) B is a smooth projective curve of genus two, $J(B)$ is the jacobian variety, and $\mathcal{M}_2 \in \operatorname{Pic}(C)$.

Proof. First we prove the "if" part. Then, clearly, X is an abelian surface.

If (X, L) is the type (1) in Theorem 2.2, then it is easy to prove that L is spanned but not very ample with $L^2 \geq 10$.

Assume that (X, L) is the type (2-1) in Therem 2.2. Then $L^2 \geq 12$. If L is not spanned, then by Theorem 1.3 there exists an effective divisor D on X such that $LD = 1$ and $D^2 = 0$. Then D is irreducible and reduced. We remark that T and $f^*\mathcal{M}_1$ are nef. Since $\deg \mathcal{M}_1 \geq 3$ and $LD = 1$, D is a fiber of f. But then $TD = TF = 2$ by assumption. Hence $LD = 2$ and this is a contradiction. Therefore L is spanned. If L is very ample, then $F \cong \mathbb{P}^1$ since $LF = 2$. Therefore L is not very ample.

Assume that (X, L) is the type (2-2) in Therem 2.2. Then $L^2 \geq 10$. If L is not spanned, then by Theorem 1.3, there exists an effective divisor D on X such that $LD = 1$ and $D^2 = 0$. But by Lemma 1.11, we get $AD \geq 2$. Therefore $LD \geq 2$ and this is a contradiction. Hence L is spanned. We can also prove that L is not very ample by the same argument as above.

Assume that (X, L) is the type (2-3) in Therem 2.2. Then $L^2 \geq 10$. If L is not spanned, then by Theorem 1.3, there exists an effective divisor D on X such that $LD = 1$ and $D^2 = 0$. Then D is irreducible and reduced. Since $LD = 1$ and $p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2$ is ample, we get $(p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2)D = 1$ and $(f^*\mathcal{M}_3)D = 0$. In particular, D is a fiber of p_1 or p_2. Hence by hypothesis, $DF > 0$ for a fiber F of f. But this is a contradiction because $(f^*\mathcal{M}_3)D = 0$ with $\deg \mathcal{M}_3 \geq 2$. Hence L is spanned. We can also prove that L is not very ample by the same argument as above.

Next we prove the "only if" part.

Step 1. *X is a double covering of a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ over a smooth elliptic curve C whose branch locus is smooth and linearly equivalent to $-2K_{\mathbb{P}(\mathcal{E})}$, and $L = \pi^*(H(\mathcal{E}))$, where $\mathcal{E}$ is an ample spanned vector bundle of rank 2 on C with* $\deg \mathcal{E} \geq 5$, *$p : \mathbb{P}(\mathcal{E}) \to C$ is the natural projection, $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}(\mathcal{E})$, and $\pi : X \to \mathbb{P}(\mathcal{E})$ is the double covering.*

Proof of Step 1. Let (X, L) be a polarized abelian surface with $L^2 \geq 10$. Assume that L is spanned but not very ample. Then by Theorem 1.3, there exists an effective divisor D on X such that one of the following holds;

(1) $LD = 1$ and $D^2 = 0$,
(2) $LD = 2$ and $D^2 = 0$.

(We remark that D^2 is even because X is an abelian surface.)
If $LD = 1$, then $D \cong \mathbb{P}^1$ since L is spanned by assumption. But this is a contradiction because X is an abelian surface. So we may assume that $LD = 2$ and $D^2 = 0$. Then by Lemma 1.6, D is a smooth elliptic curve. By Lemma 1.7, there exists an elliptic fibration $f : X \to C$ such that C is a smooth elliptic curve, any fiber of f is smooth, and D is a fiber of f. Since $LF_f = LD = 2$ for any fiber F_f of f, there exists a surjective map

$$f^* \circ f_*(L) \to L.$$

We remark that $f_*(L)$ is a locally free sheaf of rank 2 on C. Hence there exists a morphism $\pi : X \to \mathbb{P}(f_*(L))$ such that $f = p \circ \pi$, where $p : \mathbb{P}(f_*(L)) \to C$ is the natural projection. By construction π is a double covering. We put $\mathcal{E} := f_*(L)$. Let B be the branch locus of π. Then B is smooth since X is smooth. By Lemma

1.9, $B \in |2Z|$ for some $Z \in \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$ and $K_X = \pi^*(K_{\mathbb{P}(\mathcal{E})} + Z)$. We remark that by Lemma 1.9

$$\begin{aligned} h^0(K_X) &= h^0(K_{\mathbb{P}(\mathcal{E})} + Z) + h^0(K_{\mathbb{P}(\mathcal{E})}) \\ &= h^0(K_{\mathbb{P}(\mathcal{E})} + Z). \end{aligned}$$

Hence $h^0(K_{\mathbb{P}(\mathcal{E})} + Z) = 1$. Since $K_X = \mathcal{O}_X$, we get $K_{\mathbb{P}(\mathcal{E})} + Z \equiv 0$. Therefore we obtain that $K_{\mathbb{P}(\mathcal{E})} + Z = \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ since $h^0(K_{\mathbb{P}(\mathcal{E})} + Z) = 1$. So we have $B \in |-2K_{\mathbb{P}(\mathcal{E})}|$. By construction $L = \pi^*(H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}(\mathcal{E})$. Since π is finite, $H(\mathcal{E})$ is ample and so is $\mathcal{E}$. We have $\deg \mathcal{E} = H(\mathcal{E})^2 \geq 5$. We remark that by Lemma 1.9

$$\begin{aligned} h^0(L) &= h^0(H(\mathcal{E})) + h^0(H(\mathcal{E}) + K_{\mathbb{P}(\mathcal{E})}) \\ &= h^0(H(\mathcal{E})). \end{aligned}$$

Hence we obtain that $H(\mathcal{E})$ is spanned since $L = \pi^*(H(\mathcal{E}))$ and L is spanned.

This completes the proof of Step 1. $\square$

Step 2. Here we study the existence of a smooth member of $|-2K_{\mathbb{P}(\mathcal{E})}|$.

Proposition 2.3. *Let $\mathcal{E}$ be a vector bundle of rank 2 with $\deg \mathcal{E} \geq 5$ on a smooth elliptic curve C over k. Let $\mathbb{P}(\mathcal{E})$ be the projective bundle.*
(α) Assume that $\mathrm{char}\, k = 0$. Then there exists a smooth member of $|-2K_{\mathbb{P}(\mathcal{E})}|$ if and only if there exists a vector bundle $\mathcal{E}'$ on C and a line bundle $\mathcal{M}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{M}$, and $\mathcal{E}'$ and $\mathcal{M}$ satisfy one of the following three types;

(A) *$\mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{O}_C$ and $\deg \mathcal{M} \geq 3$,*
(B) *$\mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{L}_1$ and $\deg \mathcal{M} \geq 3$, where $\mathcal{L}_1 \in \mathrm{Pic}(C)$ with $\mathcal{L}_1 \not\cong \mathcal{O}_C$ and $2\mathcal{L}_1 \cong \mathcal{O}_C$,*
(C) *there exists a nontrivial extension*

$$0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{L}_2 \to 0$$

and $\deg \mathcal{M} \geq 2$, where $\mathcal{L}_2 \in \mathrm{Pic}(C)$ with $\deg \mathcal{L}_2 = 1$.

(β) Assume that $\mathrm{char}\, k > 0$. If there exists a smooth member of $|-2K_{\mathbb{P}(\mathcal{E})}|$, then there exists a vector bundle $\mathcal{E}'$ on C and a line bundle $\mathcal{M}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{M}$, and $\mathcal{E}'$ and $\mathcal{M}$ satisfy one of the above three types.

Proof. (α) Assume that $\mathrm{char}\, k = 0$. By Proposition III.15 (ii) in [6], there exist a vector bundle $\mathcal{E}'$ of rank 2 on C and a line bundle $\mathcal{M}$ on C such that $\mathcal{E} = \mathcal{E}' \otimes \mathcal{M}$ and $\mathcal{E}'$ satisfies one of the following three types;

(1) $\mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{L}_1$, where $\mathcal{L}_1 \in \mathrm{Pic}(C)$ with $\deg \mathcal{L}_1 \leq 0$.
(2) There exists a nontrivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{O}_C \to 0.$$

(3) There exists a nontrivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{L}_2 \to 0,$$

where $\mathcal{L}_2 \in \mathrm{Pic}(C)$ with $\deg \mathcal{L}_2 = 1$.

It suffices to consider the existence of a smooth member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$. Let $p : \mathbb{P}(\mathcal{E}') \to C$ be the natural projection and $H(\mathcal{E}')$ the tautological line bundle of $\mathbb{P}(\mathcal{E}')$.
(I) The case in which $\mathcal{E}'$ satisfies (1).
(I-i) The case in which $\deg \mathcal{L}_1 < 0$.
Then $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}') - 2p^*\mathcal{L}_1$ and $h^0(-K_{\mathbb{P}(\mathcal{E}')}) > 0$. Let $C_0 \in |H(\mathcal{E}')|$. Then

$$\begin{aligned}
-2K_{\mathbb{P}(\mathcal{E}')}C_0 &= (4H(\mathcal{E}') - 2p^*\mathcal{L}_1)C_0 \\
&= 4\deg \mathcal{L}_1 - 2\deg \mathcal{L}_1 \\
&= 2\deg \mathcal{L}_1 \\
&< 0.
\end{aligned}$$

Hence C_0 is a fixed component of $|-2K_{\mathbb{P}(\mathcal{E}')}|$.

Next we have

$$\begin{aligned}
(-2K_{\mathbb{P}(\mathcal{E}')} - C_0)C_0 &= (3H(\mathcal{E}') - 2p^*\mathcal{L}_1)C_0 \\
&= 3\deg \mathcal{L}_1 - 2\deg \mathcal{L}_1 \\
&= \deg \mathcal{L}_1 \\
&< 0.
\end{aligned}$$

Hence C_0 is a fixed component of $|-2K_{\mathbb{P}(\mathcal{E}')} - C_0|$. Therefore $2C_0$ is contained in the fixed part of $|-2K_{\mathbb{P}(\mathcal{E}')}|$. Hence there does not exist any smooth member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$.
(I-ii) The case in which $\deg \mathcal{L}_1 = 0$.

If $\mathcal{L}_1 \cong \mathcal{O}_C$, then $\mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{O}_C$ and $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}')$. Since $\mathcal{E}'$ is spanned, we obtain that $4H(\mathcal{E}')$ is spanned and there exists a smooth member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$. In this case, $\deg \mathcal{M} \geq 3$ since $\deg \mathcal{E} \geq 5$. This is the type (A).

Next we consider the case in which $\mathcal{L}_1 \not\cong \mathcal{O}_C$. Then $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}') - 2p^*(\mathcal{L}_1)$. So we obtain that

$$\begin{aligned}
h^0(-2K_{\mathbb{P}(\mathcal{E}')}) &= h^0(4H(\mathcal{E}') - 2p^*(\mathcal{L}_1)) \\
&= h^0(S^4(\mathcal{E}') - 2\mathcal{L}_1) \\
&= h^0(-2\mathcal{L}_1) + h^0(-\mathcal{L}_1) + h^0(\mathcal{O}_C) + h^0(\mathcal{L}_1) + h^0(2\mathcal{L}_1).
\end{aligned}$$

If $2\mathcal{L}_1 \not\cong \mathcal{O}_C$, then $h^0(\mathcal{L}_1) = h^0(-\mathcal{L}_1) = 0$ and $h^0(2\mathcal{L}_1) = h^0(-2\mathcal{L}_1) = 0$. Hence $h^0(-2K_{\mathbb{P}(\mathcal{E}')}) = 1$. On the other hand,

$$\begin{aligned}
h^0(-K_{\mathbb{P}(\mathcal{E}')}) &= h^0(2H(\mathcal{E}') - p^*(\mathcal{L}_1)) \\
&= h^0(S^2(\mathcal{E}') - \mathcal{L}_1) \\
&= h^0(-\mathcal{L}_1) + h^0(\mathcal{O}_C) + h^0(\mathcal{L}_1) \\
&= 1.
\end{aligned}$$

Hence the member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$ is not smooth.

So we assume that $2\mathcal{L}_1 \cong \mathcal{O}_C$. Then $h^0(-2K_{\mathbb{P}(\mathcal{E}')}) = 3$.

Claim 2.4. *$|-2K_{\mathbb{P}(\mathcal{E}')}|$ has no fixed part.*

Proof. Let $C_0 \in |H(\mathcal{E}')|$. Then any irreducible curve on $\mathbb{P}(\mathcal{E}')$ is numerically equivalent to $aC_0 + bF$ with $a \geq 0$ and $b \geq 0$. In particular, there exists no fiber of p which is contained in a member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$ because $\deg \mathcal{L}_1 = 0$. Let Z be the fixed part of $|-2K_{\mathbb{P}(\mathcal{E}')}|$. Assume that $Z \neq 0$. Then $Z = \alpha H(\mathcal{E}') + p^*(\mathcal{N})$, where $1 \leq \alpha \leq 3$ and $\mathcal{N} \in \mathrm{Pic}(C)$ with $\deg \mathcal{N} = 0$. Here we calculate $h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z)$. If $\alpha = 3$, then

$$h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z) = h^0(-\mathcal{L}_1 - \mathcal{N}) + h^0(-\mathcal{N}).$$

If $\alpha = 2$, then we have

$$h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z) = h^0(-\mathcal{L}_1 - \mathcal{N}) + 2h^0(-\mathcal{N}).$$

If $\alpha = 1$, then we obtain

$$h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z) = 2h^0(-\mathcal{L}_1 - \mathcal{N}) + 2h^0(-\mathcal{N}).$$

If $\mathcal{N} \cong \mathcal{O}_C$, then $h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z) \leq 2$ for $1 \leq \alpha \leq 3$ and this is a contradiction. If $\mathcal{N} \not\cong \mathcal{O}_C$ and $\mathcal{L}_1 \otimes \mathcal{N} \cong \mathcal{O}_C$, then $h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z) \leq 2$ for $1 \leq \alpha \leq 3$ and this is a contradiction. If $\mathcal{N} \not\cong \mathcal{O}_C$ and $\mathcal{L}_1 \otimes \mathcal{N} \not\cong \mathcal{O}_C$, then $h^0(-2K_{\mathbb{P}(\mathcal{E}')} - Z) = 0$ for $1 \leq \alpha \leq 3$ and this is a contradiction. Hence this completes the proof of Claim 2.4. □

Hence $|-2K_{\mathbb{P}(\mathcal{E}')}|$ is base point free since $(-2K_{\mathbb{P}(\mathcal{E}')})^2 = 0$. Hence there exists a smooth member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$ by Bertini's Theorem. In this case, $\deg \mathcal{M} \geq 3$ since $\deg \mathcal{E} \geq 5$. This is the type (B).

(II) The case in which $\mathcal{E}'$ satisfies (2).

Then $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}')$. Here we prove that $h^0(4H(\mathcal{E}')) = 1$. Assume that $h^0(4H(\mathcal{E}')) \geq 2$. Let C_0 be a section of $p : \mathbb{P}(\mathcal{E}') \to C$ such that $H(\mathcal{E}') \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(C_0)$. In particular $C_0^2 = 0$ in this case. If D is an irreducible and reduced curve on $\mathbb{P}(\mathcal{E}')$, then $D \equiv aC_0 + bF_p$ with $a \geq 0$ and $b \geq 0$, where F_p is a fiber of p. Therefore any irreducible component of $|4H(\mathcal{E}')|$ is numerically equivalent to tC_0 with $t \in \mathbb{N}$. Let $|M|$ be the movable part of $|4H(\mathcal{E}')|$. Then there exists a curve B such that B is an irreducible component of a member of $|M|$, $B \neq C_0$, and $B \equiv tC_0$ for some $t \in \mathbb{N}$. In particular, $BC_0 = 0$. Let $\pi := p|_B : B \to C$. Then there exists a morphism $\pi' : \mathbb{P}(\pi^*\mathcal{E}') \to \mathbb{P}(\mathcal{E}')$ such that $\pi \circ p' = p \circ \pi'$, where $p' : \mathbb{P}(\pi^*\mathcal{E}') \to B$ is the natural projection. Since $\mathbb{P}(\pi^*\mathcal{E}')$ has two disjoint sections, we obtain that the vector bundle $\pi^*\mathcal{E}'$ is decomposable. Hence $\mathcal{E}'$ is decomposable. But this is a contradiction by hypothesis. Therefore $h^0(4H(\mathcal{E}')) = 1$. But since $h^0(H(\mathcal{E}')) = 1$ and $h^0(4H(\mathcal{E}')) = 1$, the member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$ is not smooth.

(III) The case in which $\mathcal{E}'$ satisfies (3).

Then by p.451 in [1], the projective bundle $\mathbb{P}(\mathcal{E}')$ over C is isomorphic to $S^2(C)$, where $S^2(C)$ is the 2-fold symmetric product of C. In this case, we use the same argument as that in (0.15) in [8]. Let $V = C \times C$. Let $\pi : V \to S^2(C)$ be the quotient morphism, $\phi : C \to \mathbb{P}^1$ a morphism defined by a linear system $|\mathcal{O}_C(2x)|$ on C for some $x \in C$ and let $\{x_0, x_1, x_2, x_3\}$ be the set of ramification points of

ϕ. We take x_0 as the origin of C. Then there exists a morphism $\tau : S^2(C) \to \mathbb{P}^1$ such that $\tau \circ \pi = \phi \circ s$, where $s : V \to C$ is the difference map $(x, y) \mapsto x - y$. Furthermore τ is an elliptic fibration. Then singular fibers of τ are exactly the 3 fibers over $\{y_1, y_2, y_3\}$, where $y_i = \phi(x_i)$ for $i = 1, 2, 3$. Moreover $\tau^{-1}(y_i) = 2F_i$ for a smooth elliptic curve F_i. By the canonical bundle formula, we get

$$K_{S^2(C)} = \tau^*\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{S^2(C)}(F_1) \otimes \mathcal{O}_{S^2(C)}(F_2) \otimes \mathcal{O}_{S^2(C)}(F_3).$$

Hence $-2K_{S^2(C)} = \tau^*\mathcal{O}_{\mathbb{P}^1}(1)$ is spanned. Therefore there exists a smooth member of $|-2K_{\mathbb{P}(\mathcal{E}')}|$ by Bertini's Theorem. In this case, $\deg \mathcal{M} \geq 2$ because $\deg \mathcal{E} \geq 5$. This is the type (C).
(β) If $\operatorname{char} k > 0$, then by the same argument as in the case where $\operatorname{char} k = 0$ we can prove (β) in Proposition 2.3.

This completes the proof of Proposition 2.3. □

Step 3. By using the results of Step 1 and Step 2, we prove the "only if" part of Theorem 2.2. We use the notations in Step 1 and Step 2. By Proposition 2.3, if there exists a smooth member of $|-2K_{\mathbb{P}(\mathcal{E})}|$, then there exists a vector bundle $\mathcal{E}'$ on C and a line bundle $\mathcal{M}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{M}$, and $\mathcal{E}'$ and $\mathcal{M}$ satisfy one of the three types in Proposition 2.3.

Let $\iota : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}')$ be the isomorphism such that $p = p' \circ \iota$. Let $\pi' = \iota \circ \pi$.
(1) The case in which $\mathcal{E}'$ is the type (A) or (B) in Proposition 2.3.
Let $C_0 \in |H(\mathcal{E}')|$. Then C_0 is an irreducible reduced curve by Proposition 2.8 in [18]. Let B be a branch locus of π'. We remark that $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}')$. Since $-2K_{\mathbb{P}(\mathcal{E}')}C_0 = 0$, we get $C_0 \subset B$ or $C_0 \cap B = \emptyset$.
(1-1) The case in which $C_0 \subset B$.
Then $(\pi')^*(C_0) = 2B_1$. Since $LF = 2$ for a fiber F of f and B_1 is not contained in a fiber of f, B_1 is a section of f. Hence by Lemma 1.8, $X \cong C \times F$ and f is identified with the first projection, and B_1 is a fiber of the second projection $h : C \times F \to F$. So we get

$$\begin{aligned} L = \pi^*H(\mathcal{E}) &\cong \pi^* \circ p^*(\mathcal{M}) \otimes (\pi')^*H(\mathcal{E}') \\ &\cong f^*\mathcal{M} \otimes (\pi')^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(C_0) \\ &\cong f^*\mathcal{M} \otimes h^*\mathcal{P}, \end{aligned}$$

where $\mathcal{P} \in \operatorname{Pic}(F)$ with $\deg \mathcal{P} = 2$. Therefore we get the type (1) in Theorem 2.2.
(1-2) The case in which $C_0 \cap B = \emptyset$. Then we obtain one of the following:
(1-2-1) $(\pi')^*(C_0) = B_2$,
(1-2-2) $(\pi')^*(C_0) = B_3 + B_4$,
where B_i is an irreducible reduced curve for $i = 2, 3, 4$ with $B_3 \neq B_4$.
First we consider the case (1-2-1).

Claim 2.5. *If $(\pi')^*(C_0) = B_2$, then $\mathcal{E}'$ is the type (B) in Proposition 2.3.*

Proof. We remark that B_2 is a smooth elliptic curve. Hence by Lemma 1.7, there exists a surjective morphism $f_1 : X \to C_1$ such that C_1 is a smooth elliptic curve and B_2 is a fiber of f_1. Hence $h^0((\pi')^*(C_0)) = h^0(B_2) = 1$. On the other hand, by Lemma 1.9, we have $h^0((\pi')^*(C_0)) = h^0(H(\mathcal{E}'))$. Hence $h^0(H(\mathcal{E}')) = 1$. Therefore $\mathcal{E}'$ is the type (B) in Proposition 2.3. This completes the proof of Claim 2.5. □

Since $L = (\pi')^*(C_0) \otimes f^*\mathcal{M}$, we obtain the type (2-1) in Theorem 2.2. We remark that $B_2F = 2$ for a fiber F of f.
Next we consider the case (1-2-2).
If $(\pi')^*(C_0) = B_3 + B_4$, then we get $B_3^2 = B_4^2 = 0$ and $B_3B_4 = 0$. Since B_3 and B_4 are not contained in a fiber of f, we get that $B_3F = B_4F = 1$ for a fiber F of f because $(\pi')^*(C_0)F = 2$. Hence B_3 and B_4 are sections of f. Hence by Lemma 1.8, we get that $X \cong C \times F$, f is identified with the first projection, and B_i is a fiber of the second projection $h : X \to F$ for $i = 3, 4$. Hence by the same argument as the case (1-1), we obtain $L = f^*\mathcal{M} \otimes h^*\mathcal{P}'$, where $\mathcal{P}' \in \mathrm{Pic}(F)$ with $\deg \mathcal{P}' = 2$. Therefore we get the type (1) in Theorem 2.2.
(2) The case in which $\mathcal{E}'$ is the type (C) in Proposition 2.3.
Then $H(\mathcal{E}')$ is ample with $H(\mathcal{E}')^2 = 1$. We put $\mathcal{O}_X(A) = (\pi')^*(H(\mathcal{E}'))$. Then $AF = 2$ for a fiber F of f. Since $A^2 = 2$ and X is an abelian surface, we obtain $g(A) = 2$ and we get that (X, A) is one of the type (1) or (2) in Proposition 1.10. We remark that $L = (\pi')^*(H(\mathcal{E}')) \otimes f^*\mathcal{M} = \mathcal{O}_X(A) \otimes f^*\mathcal{M}$. If (X, A) is the type (2) in Proposition 1.10, then any fiber of f is not contained in a fiber of p_i and any fiber of p_i is a section of f for $i = 1, 2$ since $AF = 2$. Therefore if (X, A) is the type (1) (resp. (2)) in Proposition 1.10, then we get the type (2-2) (resp. (2-3)) in Theorem 2.2. This completes the proof of Theorem 2.2. □

Remark 2.2.1.
(1) In [19], Hulek and Lange proved the same result as Step 1 in the proof of Theorem 2.2 in some special case of an abelian surface over $\mathbb{C}$.
(2) Assume that $\mathrm{char}\, k = 0$. Then the types (1), (2-1), (2-2), and (2-3) in Theorem 2.2 do really exist.

We can easily prove the existence of the type (1) in Theorem 2.2.

Next we consider the type (2-3) in Theorem 2.2. Let E be a smooth elliptic curve and let $X := E \times E$. Let D be a diagonal of X. We remark that D is a smooth elliptic curve. Then we put $L = p_1^*D_1 + p_2^*D_2 + tD$, where $p_i : E \times E \to E$ is the i-th projection, and t is an integer which is greater than 1. Then by Lemma 1.7, this (X, L) is the type (2-3) in Theorem 2.2.

Next we consider the case (2-1) in Theorem 2.2. Let C be a smooth elliptic curve. Let $\mathcal{E}' := \mathcal{O}_C \oplus \mathcal{L}_1$, where $\mathcal{L}_1 \in \mathrm{Pic}(C)$ with $\mathcal{L}_1 \not\cong \mathcal{O}_C$ and $2\mathcal{L}_1 \cong \mathcal{O}_C$. By the proof of Proposition 2.3, we obtain that $\mathrm{Bs}\,|-2K_{\mathbb{P}(\mathcal{E}')}| = \emptyset$. Let $C_0 \in |H(\mathcal{E}')|$ be an irreducible reduced curve, where $H(\mathcal{E}')$ is the tautological line bundle. We remark that $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}')$. Since $h^0(4H(\mathcal{E}')) = 3 > 2 = h^0(3H(\mathcal{E}'))$, there exists a smooth member $B \in |-2K_{\mathbb{P}(\mathcal{E}')}|$ such that $B \not\supset C_0$ by Bertini's Theorem. Hence $C_0 \cap B = \emptyset$. Let $\pi' : X \to \mathbb{P}(\mathcal{E}')$ be the double covering whose branch locus is B. By Lemma 1.9, we have

$$\begin{aligned}
h^0((\pi')^*H(\mathcal{E}')) &= h^0(H(\mathcal{E}')) + h^0(H(\mathcal{E}') + K_{\mathbb{P}(\mathcal{E}')}) \\
&= h^0(H(\mathcal{E}')) \\
&= 1.
\end{aligned}$$

Hence $h^0((\pi')^*(C_0)) = h^0((\pi')^*(H(\mathcal{E}'))) = 1$. Assume that $(\pi')^*(C_0) = B_3 + B_4$, where B_3 and B_4 are irreducible reduced curves with $B_3 \neq B_4$. We remark that $B_3^2 = B_4^2 = 0$ and $B_3B_4 = 0$. We put $f := p' \circ \pi'$, where $p' : \mathbb{P}(\mathcal{E}') \to C$ is the natural projection. Then since $B_3F = B_4F = 1$ for a fiber F of f, we obtain that $X \cong C \times F$, f is the first projection via this isomorphism, and B_3 and B_4 are fibers of the second projection $h : X \to F$. Hence $B_3 + B_4 = h^*(\mathcal{P})$ for $\mathcal{P} \in \mathrm{Pic}(F)$ with $\deg \mathcal{P} = 2$. So we have $h^0((\pi')^*(C_0)) = h^0(B_3 + B_4) = h^0(h^*\mathcal{P}) = 2$ since F is a smooth elliptic curve. This is a contradiction. Hence $(\pi')^*(C_0)$ is an irreducible reduced curve. We put $L = (\pi')^*H(\mathcal{E}')\otimes f^*\mathcal{M}$, where $\mathcal{M} \in \mathrm{Pic}(C)$ with $\deg \mathcal{M} \geq 3$. Then this (X, L) is the type (2-1) in Theorem 2.2.

Next we consider the type (2-2) in Theorem 2.2. Let C be a smooth elliptic curve and let $\mathcal{E}'$ be a locally free sheaf of rank 2 on C such that there exists a nontrivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{L}_2 \to 0$$

for $\mathcal{L}_2 \in \mathrm{Pic}(C)$ with $\deg \mathcal{L}_2 = 1$. Then by the proof of Proposition 2.3, we obtain that $\mathrm{Bs}\,|-2K_{\mathbb{P}(\mathcal{E}')}| = \emptyset$. Let C_0 be an irreducible reduced curve such that $C_0 \in |H(\mathcal{E}')|$. We remark that $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}') - (p')^*(2\det \mathcal{E}')$, and $\deg \mathcal{E}' = 1$, where $p' : \mathbb{P}(\mathcal{E}') \to C$ is the natural projection. By Theorem 1.2 in [13], we get $h^0(3H(\mathcal{E}') - (p')^*(2\det \mathcal{E}')) = 0$. On the other hand, we have $h^0(-2K_{\mathbb{P}(\mathcal{E}')}) = 2$ and $h^0(-2K_{\mathbb{P}(\mathcal{E}')}|_{C_0}) = 2$. Hence the map

$$H^0(-2K_{\mathbb{P}(\mathcal{E}')}) \to H^0(-2K_{\mathbb{P}(\mathcal{E}')}|_{C_0})$$

is an isomorphism. Hence, by Bertini's Theorem, there exists a smooth member B of $|-2K_{\mathbb{P}(\mathcal{E}')}|$ such that $B|_{C_0}$ is smooth. Because $BC_0 = 2$, we get $B|_{C_0} = P_1 + P_2$, where P_1 and P_2 are distinct points on C_0. Let $\pi' : X \to \mathbb{P}(\mathcal{E}')$ be the double covering whose branch locus is B. Assume that $(\pi')^*(C_0) = B_1 + B_2$ for irreducible reduced curves B_1 and B_2 with $B_1 \neq B_2$. We put $A := (\pi')^*(C_0)$. Then A is ample and $g(A) = 2$. Hence by Proposition 1.10, we get that $X \cong E_1 \times E_2$ and B_i is a fiber of the i-th projection for $i = 1, 2$. In particular, $B_1B_2 = 1$. Hence $B_1 \cap B_2$ is a point. But since $B|_{C_0} = P_1 + P_2$, we get $\sharp(B_1 \cap B_2) \geq 2$. This is a contradiction. Hence A is an irreducible reduced curve. Therefore by Proposition 1.10, we get that $X \cong J(B)$ and A is a translation of B on $J(B)$, where B is a smooth projective curve of genus two and $J(B)$ is the jacobian varieties. We put $L = \mathcal{O}_X(A) \otimes (p' \circ \pi')^*\mathcal{M}$ for $\mathcal{M} \in \mathrm{Pic}(C)$ with $\deg \mathcal{M} \geq 2$. This (X, L) is the type (2-2) in Theorem 2.2.

Theorem 2.6. *Let (X, L) be a polarized surface over $\mathbb{C}$. Then X is an abelian surface and L is very ample but not 2-very ample with $L^2 \geq 14$ if and only if X is a smooth irreducible divisor on a $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E})$ over a smooth elliptic curve C such that $X \in |-K_{\mathbb{P}(\mathcal{E})}|$ and $f := p|_X : X \to C$ is surjective with connected fibers, and $L = H(\mathcal{E})|_X$, where $\mathcal{E}$ is a locally free sheaf of rank 3 on C with $7 \leq e := \deg \mathcal{E}$ and $h^0(\mathcal{E}) = e$, $H(\mathcal{E})$ is the tautological line bundle, and $p : \mathbb{P}(\mathcal{E}) \to C$ is the natural projection.*

Proof. First we prove the "if" part. By assumption, X is smooth, $f : X \to C$ is a surjective morphism with connected fibers, and $K_X = (K_{\mathbb{P}(\mathcal{E})} + X)|_X = \mathcal{O}_X$. Hence $p_g(X) = 1$. Since $g(C) = 1$, we have $q(X) = 2$ by the classification theory of surfaces. Therefore X is an abelian surface. Next we prove that L is ample. We remark that $h^0(H(\mathcal{E})) = h^0(\mathcal{E}) = e \geq 7$. If $X \subset D$ for some $D \in |H(\mathcal{E})|$, then $D - X \geq 0$. But since $D - X = H(\mathcal{E}) + K_{\mathbb{P}(\mathcal{E})} = -2H(\mathcal{E}) + p^*(\det \mathcal{E})$, a divisor

$(D-X)|_{F_p} = -2H(\mathcal{E})|_{F_p}$ is not effective for a general fiber F_p of p. This is a contradiction. Hence $X \not\subset D$ for any $D \in |H(\mathcal{E})|$ and $D|_X$ is an effective divisor on X. On the other hand, $(D|_X)^2 = 2e \geq 14$. Hence $D|_X$, that is, L is ample.

Next we prove that L is very ample. If L is not very ample, then by Theorem 2.1 and Step 1 in the proof of Theorem 2.2, (X, L) is one of the following types;
(A) $X \cong E_1 \times E_2$ and $L = p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2$, where E_i is a smooth elliptic curve, $p_i : X \to E_i$ is the i-th projection and $\mathcal{D}_i \in \mathrm{Pic}(E_i)$ for $i = 1, 2$ with $\deg \mathcal{D}_1 = 1$ and $\deg \mathcal{D}_2 \geq 7$.
(B) X is a double covering of a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{F})$ over a smooth elliptic curve E whose branch locus is smooth and linearly equivalent to $-2K_{\mathbb{P}(\mathcal{F})}$, and $L = \pi^*(H(\mathcal{F}))$, where $\mathcal{F}$ is an ample spanned vector bundle of rank 2 on E with $\deg \mathcal{F} \geq 7$, $r : \mathbb{P}(\mathcal{F}) \to E$ is the natural projection, and $\pi : X \to \mathbb{P}(\mathcal{F})$ is the double covering.

Let F be a fiber of f. Then we remark that

$$\begin{aligned} LF &= (H(\mathcal{E})|_X)(F_p|_X) \\ &= (H(\mathcal{E}))(-K_{\mathbb{P}(\mathcal{E})})(F_p) \\ &= (H(\mathcal{E}))(3H(\mathcal{E}) - p^*(\det(\mathcal{E})))(F_p) \\ &= 3. \end{aligned}$$

(1) The case in which (X, L) is the type (A).
Since $LF = 3$ and $L = p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2$ with $\deg \mathcal{D}_1 = 1$ and $\deg \mathcal{D}_2 \geq 7$, any fiber of f is contained in a fiber of p_2. Hence there exists a morphism $\delta : C \to E_2$ such that $p_2 = \delta \circ f$. Since f and p_2 have connected fibers, δ is an isomorphism. But this is impossible because $LF = 3$ and $LF_2 = 1$, where F_2 is a fiber of p_2.
(2) The case in which (X, L) is the type (B).
Let $h : X \to E$ be the morphism $r \circ \pi$. Then h has connected fibers. Let F_h be a fiber of h. If $F_h F > 0$, then $(F_h + F)^2 \geq 2$. Since $L^2 \geq 14$, we obtain $(L(F_h + F))^2 \geq L^2(F_h + F)^2 \geq 28$ by Hodge index Theorem. But this is a contradiction because $L(F_h + F) = 5$. Hence $F_h F = 0$ and any fiber F of f is contained in a fiber of h. So there exists a morphism $\delta' : C \to E$ such that $h = \delta' \circ f$. Since f and h have connected fibers, δ' is an isomorphism. But this is impossible becuse $LF = 3$ and $LF_h = 2$.
By the above argument (1) and (2), we obtain that L is very ample. Since $LF = 3$, L is not 2-very ample by Theorem 1.5.

Next we prove the "only if" part. Let X be an abelian surface and let L be very ample but not 2-very ample with $L^2 \geq 14$. By Theorem 1.3, if L is not 2-very ample, then there exists an effective divisor D on X such that

$$LD - 3 \leq D^2 < \frac{LD}{2} < 3.$$

We remark that D^2 is even because X is an abelian surface. So the following is possible.

(1) $LD = 5$ and $D^2 = 2$,
(2) $LD = 3$ and $D^2 = 0$.

If $LD = 5$ and $D^2 = 2$, then $(LD)^2 = 25 < 28 \leq (L^2)(D^2)$. This is impossible by Hodge index Theorem. So D is an effective divisor with $LD = 3$ and $D^2 = 0$. By Lemma 1.6, D is a smooth elliptic curve. By Lemma 1.7, there exists an elliptic fibration $f : X \to C$ such that any fiber of f is smooth and D is a fiber of f, where C is a smooth elliptic curve. Since $LF = LD = 3$, there exists a surjective morphism

$$f^* \circ f_*(L) \to L.$$

Hence there exists a morphism $\iota : X \to \mathbb{P}(\mathcal{E})$ such that $f = p \circ \iota$, where $\mathcal{E} := h_*(L)$ is a locally free sheaf of rank 3 on C and $p : \mathbb{P}(\mathcal{E}) \to C$ is the projection. We put $e := \deg \mathcal{E}$. By construction ι is an embedding and $L = H(\mathcal{E})|_X$. Since $\mathcal{O}_X = K_X = (K_{\mathbb{P}(\mathcal{E})} + X)|_X$, we obtain that $X \in |3H(\mathcal{E}) + p^*(\mathcal{D})|$ for some $\mathcal{D} \in \mathrm{Pic}(C)$. Since

$$\begin{aligned} f^*(\mathcal{D} + \det \mathcal{E}) &\cong p^*(\mathcal{D} + \det \mathcal{E})|_X \\ &= \mathcal{O}_X, \end{aligned}$$

we obtain $\mathcal{D} + \det \mathcal{E} \cong \mathcal{O}_C$. Therefore $X \in |3H(\mathcal{E}) - p^*(\det \mathcal{E})| = |-K_{\mathbb{P}(\mathcal{E})}|$. Furthermore we have

$$\begin{aligned} 14 \leq L^2 &= (H(\mathcal{E})|_X)^2 \\ &= H(\mathcal{E})^2(3H(\mathcal{E}) - p^*(\det \mathcal{E})) \\ &= 2e, \end{aligned}$$

and

$$h^0(\mathcal{E}) = h^0(f_*(L)) = h^0(L) = \frac{L^2}{2} = e.$$

This completes the proof of Theorem 2.6. $\square$

By Theorem 2.6, it is necessary to study a vector bundle $\mathcal{E}$ of rank three on a smooth elliptic curve such that $|-K_{\mathbb{P}(\mathcal{E})}|$ has a smooth irreducible divisor. We hope this will be treated in a future paper.

Theorem 2.7. *Let (X, L) be a polarized surface over $\mathbb{C}$. Then X is an abelian surface and L is 2-very ample but not 3-very ample with $L^2 \geq 18$ if and only if*
(A) *$X \cong J(B)$ for some smooth curve B of genus 2 and $L \equiv 3B$, where $J(B)$ is the Jacobian variety of B, or*
(B) *(X, L) satisfies* (B-1) *and* (B-2)*:*
(B-1) *X is a smooth subvariety of a $\mathbb{P}^3$-bundle $\mathbb{P}(\mathcal{E})$ over a smooth elliptic curve C and $L = H(\mathcal{E})|_X$ such that $f := p|_X$ is surjective with connected fibers, any fiber F of f is smooth, F is a complete intersection of two quadrics in $\mathbb{P}^3$, and the conormal sheaf of X in $\mathbb{P}(\mathcal{E})$ is $f^*\mathcal{F} \otimes \mathcal{O}(-2L)$, where $\mathcal{E}$ is a vector bundle of rank 4 on C with $e := \deg \mathcal{E} \geq 9$ and $h^0(\mathcal{E}) = e$, $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}(\mathcal{E})$, $p : \mathbb{P}(\mathcal{E}) \to C$ is the natural projection, and $\mathcal{F}$ is a vector bundle of rank 2 on C with $\det \mathcal{E} = \det \mathcal{F}$.*
(B-2) *$(X, L) \not\cong (E_1 \times E_2, p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2 \otimes \mathcal{O}_X(S))$ with $1 \leq \deg \mathcal{D}_1 \leq 4$ and $\deg \mathcal{D}_2 = 2$, where E_1 and E_2 are smooth elliptic curves, $p_i : E_1 \times E_2 \to E_i$ is the i-th projection and $\mathcal{D}_i \in \mathrm{Pic}(E_i)$ for $i = 1, 2$, and S is a section of p_1 with $\deg \mathcal{D}_1 + SE_1 = 4$.*

Proof. First we prove the "if" part. We consider the case in which $X \cong J(B)$ for some smooth curve B of genus 2 and $L \equiv 3B$. Hence L is 2-very ample by Theorem

1.5 and Lemma 1.11. Furthermore by Theorem 1.5, L is not 3-very ample because $LB = 6$ and $g(B) = 2$.

Next we consider the case in which (X, L) is the type (B). We remark that f is surjective with connected fibers. By assumption we have

$$\begin{aligned} K_X &= K_{\mathbb{P}(\mathcal{E})}|_X - f^* \det \mathcal{F} + 4L \\ &= f^*(\det \mathcal{E} - \det \mathcal{F}) \\ &= \mathcal{O}_X. \end{aligned}$$

Since $g(C) = 1$, we have $q(X) = 2$ by the classification theory of surfaces. Hence X is an abelian surface. We remark that $f_*(L) = \mathcal{E}$. Hence $h^0(\mathcal{E}) = h^0(L)$. By Leray spectral sequence we obtain $h^1(L) = h^1(f_*(L))$ since $h^1(L_F) = 0$ for any fiber F of f. By Riemann-Roch Theorem for $\mathcal{E}$ on C, we have

$$\begin{aligned} h^0(L) = h^0(\mathcal{E}) &= h^1(\mathcal{E}) + \operatorname{rank}(\mathcal{E})(1 - g(C)) + \deg \mathcal{E} \\ &\geq \deg \mathcal{E} \\ &\geq 9. \end{aligned}$$

Since X is an abelian surface and $h^0(L) \geq 9$, we obtain $L^2 \geq 0$. Assume that $L^2 = 0$. Then by Riemann-Roch Theorem for L on X, we have $\chi(L) = 0$. Since $h^0(L) \geq 9$, we obtain $h^2(L) = 0$. Therefore $h^0(L) = h^1(L)$. But since $h^0(\mathcal{E}) - h^1(\mathcal{E}) = h^0(L) - h^1(L)$, we obtain $\deg \mathcal{E} = \chi(\mathcal{E}) = 0$ by Riemann-Roch Theorem for $\mathcal{E}$ on C. This is a contradiction by assumption. Hence $L^2 > 0$. Therefore L is ample. By Riemann-Roch Theorem for L on X, we have $L^2 = 2h^0(L)$. Since $h^1(\mathcal{E}) = h^1(L) = 0$, we obtain $h^0(\mathcal{E}) = \deg \mathcal{E}$. Hence $L^2 = 2 \deg \mathcal{E} \geq 18$. Next we will prove that L is 2-very ample. Assume that L is not 2-very ample. Then by Theorem 2.1, Theorem 2.2, and Theorem 2.6, there exists a fiber space $h : X \to C'$ such that $1 \leq LF_h \leq 3$, where C' is a smooth elliptic curve and F_h is a fiber of h.

Assume that $F_f F_h > 0$ for a fiber F_f of f.

Claim 2.8. *If $F_f F_h = 1$, then $X \cong F_f \times F_h$ and $L \cong p_f^* \mathcal{D}_f \otimes p_h^* \mathcal{D}_h \otimes \mathcal{O}_X(S)$ with $1 \leq \deg \mathcal{D}_f \leq 4$ and $\deg \mathcal{D}_h = 2$, where p_f (resp. p_h) is the projection $F_f \times F_h \to F_f$ (resp. $F_f \times F_h \to F_h$), $\mathcal{D}_f \in \operatorname{Pic}(F_f)$, $\mathcal{D}_h \in \operatorname{Pic}(F_h)$, and S is a section of p_f with $\deg \mathcal{D}_f + SF_f = 4$.*

Proof. Since $F_f F_h = 1$, we can easily prove that $X \cong F_f \times F_h$. If $LF_h \leq 2$, then $L(F_f + F_h) \leq 6$. Since $L^2 \geq 18$ and $(F_f + F_h)^2 = 2$, we obtain that $L \equiv 3(F_f + F_h)$ by Hodge index Theorem. But then $LF_f = 3$. This is a contradiction. Hence $LF_h = 3$. We put $d' = L^2/2$. Since $L(F_f + F_h) = 7$, we obtain $L^2 \leq 24$ by Hodge index Theorem. So we have $9 \leq d' \leq 12$. On the other hand, we get $h^0(L - 2F_f) > 0$. Since $(L - 2F_f)F_h = 1$, there exist a section S of p_f, $\mathcal{B}_f \in \operatorname{Pic}(F_f)$, and $\mathcal{B}_h \in \operatorname{Pic}(F_h)$ with $\deg \mathcal{B}_h = 2$ such that $L = p_f^* \mathcal{B}_f \otimes p_h^* \mathcal{B}_h \otimes \mathcal{O}_X(S)$. Let $a = \deg \mathcal{B}_f$ and $b = SF_f$. We remark that $S^2 = 0$. Since $L^2 = 2d'$ and $LF_f = 4$, we obtain $3a + 2b = d'$ and $a + b = 4$. So we have $a = d' - 8$. Since $9 \leq d' \leq 12$, we obtain $(d', a, b) = (9, 1, 3), (10, 2, 2), (11, 3, 1)$, and $(12, 4, 0)$. This completes the proof of Claim 2.8 by putting $\mathcal{D}_f = \mathcal{B}_f$ and $\mathcal{D}_h = \mathcal{B}_h$. $\square$

But by assumption (B-2), $F_f F_h = 1$ case cannot occur. So we have $F_f F_h \geq 2$. By assumption we have $LF_f = 4$. Since $L(F_f + F_h) \leq 7$ and $L^2 \geq 18$, we obtain

$(L(F_f+F_h))^2 \leq 49 < 72 \leq L^2(F_f+F_h)^2$. But this is impossible by Hodge index Theorem.

If $F_fF_h = 0$, then $h(F_f)$ is a point for any fiber F_f of f and there exists a morphism $\delta : C \to C'$ such that $h = \delta \circ f$. Since f and h have connected fibers, δ is an isomorphism. But this cannot occur since $LF_h \leq 3$ and $LF_f = 4$. Hence L is 2-very ample.

Next we will prove that L is not 3-very ample. Since there exists an effective divisor F_f on X such that $LF_f = 4$ and $F_f^2 = 0$, L is not 3-very ample by Theorem 1.5.

Next we prove the "only if" part. Let X be an abelian surface and let L be 2-very ample but not 3-very ample with $L^2 \geq 18$. By Theorem 1.3 and Proposition 1.4, there exists an effective divisor D on X with $(LD, D^2) = (6,2)$, $(5,2)$, or $(4,0)$. (We remark that D^2 is even and X has no rational curve.)

(I) The case in which $(LD, D^2) = (5,2)$.
This case cannot occur by Hodge index Theorem.

(II) The case in which $(LD, D^2) = (6,2)$.
Then by Hodge index Theorem, we obtain $L \equiv 3D$. We remark that D is ample. Since $g(D) = 2$, by Proposition 1.10 we have

(1) $X \cong J(B)$ and $D \equiv B$, where B is a smooth curve of genus 2 and $J(B)$ is the Jacobian variety of B.
(2) $X \cong E_1 \times E_2$ and $D = F_1 + F_2$, where E_i is an elliptic curve and F_i is a fiber of $p_i : X \to E_i$ for $i = 1, 2$.

If $X \cong E_1 \times E_2$ and $D = F_1 + F_2$, then $L \equiv 3(F_1 + F_2)$. But since $LF_i = 3$, we obtain that L is not 2-very ample by Theorem 1.5. Hence this case cannot occur. So we get the type (A) in Theorem 2.7.

(III) The case in which $(LD, D^2) = (4,0)$.
By Lemma 1.6, D is a smooth ellitic curve. By Lemma 1.7, there exists an elliptic fibration $f : X \to C$ such that C is a smooth elliptic curve, any fiber of f is smooth, and D is a fiber of f. Since $LF_f = LD = 4$, we have the surjective map

$$f^* \circ f_*(L) \to L,$$

where F_f is a fiber of f. We put $\mathcal{E} = f_*(L)$. Then $\mathcal{E}$ is a vector bundle of rank 4 on C. Hence there exists a morphism $\iota : X \to \mathbb{P}(\mathcal{E})$ such that $f = p \circ \iota$, where $p : \mathbb{P}(\mathcal{E}) \to C$ is the natural projection. By construction, ι is an embedding. So X is a smooth subvariety of $\mathbb{P}(\mathcal{E})$. Let $H(\mathcal{E})$ be the tautological line bundle of $\mathbb{P}(\mathcal{E})$. Then $L = H(\mathcal{E})|_X$. Since L is ample, we obtain $h^0(L) = (1/2)(L^2) \geq 9$ and $h^1(L) = 0$. On the other hand by Leray spectral sequence we have $h^1(L) = h^1(f_*(L)) = h^1(\mathcal{E})$. Hence $h^0(\mathcal{E}) = \chi(\mathcal{E}) = \deg \mathcal{E}$ by Riemann-Roch Theorem for $\mathcal{E}$ on C. Therefore $\deg \mathcal{E} = h^0(\mathcal{E}) = h^0(L) \geq 9$. We remark that any fiber F_f of f is a complete intersection of two quadrics of $\mathbb{P}^3$ by construction. Let $\mathcal{I}$ be the ideal sheaf of X in $\mathbb{P}(\mathcal{E})$. Then there exists an exact sequence

$$0 \to \mathcal{I} \otimes 2H(\mathcal{E}) \to 2H(\mathcal{E}) \to \mathcal{O}_X(2L) \to 0.$$

Since $R^1p_*(\mathcal{I} \otimes 2H(\mathcal{E})) = 0$, the above exact sequence yields an exact sequence

$$0 \to p_*(\mathcal{I} \otimes 2H(\mathcal{E})) \to S^2(\mathcal{E}) \to f_*\mathcal{O}_X(2L) \to 0.$$

We put $\mathcal{F} = p_*(\mathcal{I} \otimes 2H(\mathcal{E}))$. We remark that $\mathcal{F}$ is a vector bundle of rank 2 on C. Let $\mathcal{C} := \mathcal{I}/\mathcal{I}^2$ be the conormal sheaf of X in $\mathbb{P}(\mathcal{E})$. Then $\mathcal{F} = f_*(\mathcal{C} \otimes \mathcal{O}_X(2L))$. So we have $\mathcal{C} = f^*\mathcal{F} \otimes \mathcal{O}_X(-2L)$. Since $K_X = K_{\mathbb{P}(\mathcal{E})}|_X - f^* \det \mathcal{F} + 4L$ and X is an abelian surface, we obtain that $\mathcal{O}_X = f^*(\det \mathcal{E} - \det \mathcal{F})$. Hence we have $\det \mathcal{E} = \det \mathcal{F}$. So we get that (X, L) satisfies (B-1).

Assume that $X \cong E_1 \times E_2$ and $L = p_1^*\mathcal{D}_1 \otimes p_2^*\mathcal{D}_2 \otimes \mathcal{O}_X(S)$ with $1 \leq \deg \mathcal{D}_1 \leq 4$, $\deg \mathcal{D}_2 = 2$, and $\deg \mathcal{D}_1 + SE_1 = 4$, where E_1 and E_2 are smooth elliptic curves, $p_i : E_1 \times E_2 \to E_i$ is the i-th projection and $\mathcal{D}_i \in \mathrm{Pic}(E_i)$ for $i = 1, 2$, and S is a section of p_1. Then $LE_2 = 3$ and L is not 2-very ample by Theorem 1.5. Hence this case is impossible. Therefore (X, L) satisfies (B-2).

This completes the proof of Theorem 2.7. $\square$

References

1. M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. (3) **7** (1957), 414–452.
2. W. Barth, C. Peters, and A. Van de Ven, *Comact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, Springer-Verlag, 1984.
3. E. Ballico and A. J. Sommese, *Projective surfaces with k-very ample line bundles of degree $\leq 4k + 4$*, Nagoya Math. J. **136** (1994), 57–79.
4. Th. Bauer and T. Szemberg, *Higher order embeddings of abelian varieties*, Math. Z. **224** (1997), 449–455.
5. Th. Bauer and T. Szemberg, *Primitive higher order embeddings of abelian surfaces*, Trans. Amer. Math. Soc. **349** (1997), 1675–1683.
6. A. Beauville, *Complex algebraic surfaces, second edition*, London Mathematical Society Student Texts, vol. 34, Cambridge University Press, 1996.
7. M. C. Beltrametti, A. Lanteri, and M. Palleschi, *Algebraic surfaces containing an ample divisor of arithmetic genus two*, Arkiv. för mat **25** (1987), 189–210.
8. M. C. Beltrametti and A. J. Sommese, *On generically polarized Gorenstein surfaces of sectional genus two*, J. Reine Angew. Math. **386** (1988), 172–186.
9. M. C. Beltrametti and A. J. Sommese, *On k-spannedness for projective surfaces*, In Algebraic Geometry (L'Aquila, 1988) Lecture Notes in Math. **1417** (1990), Springer, 24–51.
10. M. C. Beltrametti and A. J. Sommese, *Zero cycles and k-th order embeddings of smooth projective surfaces*, in Problems in the Theory of Surfaces and their Classification, Cortona, Italy, 1988, ed. by F. Catanese and C. Ciliberto, Sympos. Math., vol. 32, 1992, pp. 33–48.
11. M. C. Beltrametti and A. J. Sommese, *On the preservation of k-very ampleness under adjunction*, Math. Z. **212** (1993), 257–283.
12. M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Math. **16** (1995), Walter de Gruyter, Berlin, NewYork.
13. A. Biancofiore and E. L. Livorni, *On the genus of a hyperplane section of a geometrically ruled surface*, Ann. Mat. Pura Appl. **147** (1987), 173–185.
14. E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p, II*, Complex Analysis and Algebraic Geometry, A collection of papers dedicated to K. Kodaira (W. L. Baily, Jr. and T. Shioda, eds) (1977), Iwanami Shoten Publishers, Tokyo, and Princeton University Press, Cambridge, 23–42.
15. A. Collino, *A new proof of the Ran-Matsusaka criterion for jacobians*, Proc. Amer. Math. Soc. **92** (1984), 329–331.
16. T. Fujita, *Classification of polarized manifolds of sectional genus two*, the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata (1987), 73–98.
17. T. Fujita, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Series, vol. 155, 1990.
18. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math, vol. 52, Springer-Verlag, 1977.

19. K. Hulek and H. Lange, *Examples of Abelian surfaces in* $\mathbb{P}^4$, J. Reine Angew. Math. **363** (1985), 201–216.
20. S. Lang, *Abelian Varieties*, Springer-Verlarg, 1983.
21. H. Lange and Ch. Birkenhake, *Complex Abelian Varieties*, Grundlehren der mathematischen Wissenschaften, vol. 302, Springer-Verlag, 1992.
22. D. Mumford, *Enriques' classification of surfaces in char. p, I*, Global analysis. D. C. Spencer and S. Iyanaga eds. (1969), Tokyo University Press and Princeton University Press, 325–339.
23. D. Mumford, *Abelian varieties*, Oxford University Press, 1970.
24. T. Nakashima, *On Reider's method for surfaces in positive characteristic*, J. reine. angew. Math. **438** (1993), 175–185.
25. U. Persson, *Chern invariants of surfaces of general type*, Compositio Math. **43** (1981), 3–58.
26. S. Ramanan, *Ample divisors on Abelian surfaces*, Proc. London Math. Soc. (3) **51** (1985), 231–245.
27. I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. **127** (1988), 309–316.
28. N. I. Shepherd-Barron, *Unstable vector bundles and linear systems on surfaces in characteristic p*, Invent. Math. **106** (1991), 243–262.
29. H. Terakawa, *The k-very ampleness and k-spannedness on polarized abelian surfaces*, to appear in Math. Nachr.

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