Calculations of sectional Euler numbers and sectional Betti numbers of special polarized manifolds

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1 Introduction

In this note, we will calculate the ith sectional Euler number $e_i(X, L)$ and the ith sectional Betti number $b_i(X, L)$ of some special polarized manifolds (X, L). We also note that results in this note are useful for classifications of polarized manifolds (for example see [5]). At any time, we will update this note if we complete calculations of sectional Euler numbers and sectiona Betti numbers of new example[1].

2 Preliminaties

Notation 2.1 Let (X, L) be a polarized manifold of dimension n. For every integers i and j with $0 \leq i \leq n$ and $0 \leq j \leq i$, we put

$$C^i_j(X, L) := \sum_{l=0}^{j} (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X)L^l,$$

Definition 2.1 ([3]) Let (X, L) be a polarized manifold of dimension n, and let i and j be integers with $0 \leq j \leq i \leq n$.

(i) The *i-th sectional Euler number $e_i(X, L)$ of (X, L)* is defined by the following:

$$e_i(X, L) := C^i_i(X, L)L^{n-i}.$$

(ii) The *i-th sectional Betti number $b_i(X, L)$ of (X, L)* is defined by the following:

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0, \\ (-1)^i \left(e_i(X, L) - \sum_{j=0}^{i-1} 2(-1)^j h^j(X, \mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

Remark 2.1 (i) For every integers i and j with $0 \leq j \leq i \leq n$, $e_i(X, L)$, $b_i(X, L)$ and $w^j_i(X, L)$ are integer (see [3]).

(ii) If $i = 0$, then $e_0(X, L) = b_0(X, L) = L^n$. If $i = n$, then $e_n(X, L) = e(X)$ and $b_n(X, L) = h^n(X, \mathbb{C})$.

[1] If you find a mistake in this note, please let me know.

3 Calculations

Example 3.1 The case where (X, L) is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
Then

$$e_i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = e(\mathbb{P}^i) = i + 1$$

and

$$b_i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = b(\mathbb{P}^i) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Example 3.2 The case where (X, L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
Then

$$b_n(\mathbb{Q}^n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

$$b_{n-1}(\mathbb{Q}^n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd,} \end{cases}$$

$$b_i(\mathbb{Q}^n) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-2, \\ 0, & \text{if } i \text{ is odd with } i \leq n-2, \end{cases}$$

Hence

$$e_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) = e_i(\mathbb{Q}^i) = \begin{cases} i+2, & \text{if } i \text{ is even,} \\ i+1, & \text{if } i \text{ is odd,} \end{cases}$$

and

$$b_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) = (-1)^i \left(e_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) - 2\sum_{j=0}^{i-1} b_j(\mathbb{Q}^n) \right) = \begin{cases} 2, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Example 3.3 The case where (X, L) is $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
Set $H = \mathcal{O}_{\mathbb{P}^4}(1)$. Then $c_1(\mathbb{P}^4) = 5H$, $c_2(\mathbb{P}^4) = 10H^2$, $c_3(\mathbb{P}^4) = 10H^3$, $c_4(\mathbb{P}^4) = 5H^4 = 5$.

Hence

$$\begin{aligned}
e_0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= (2H)^4 = 16, \\
e_1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= \sum_{l=0}^{1} (-1)^l \binom{2+l}{l} c_{1-l}(X)(2H)^{3+l} = -8, \\
e_2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= \sum_{l=0}^{2} (-1)^l \binom{1+l}{l} c_{2-l}(X)(2H)^{2+l} = 8, \\
e_3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= \sum_{l=0}^{3} (-1)^l \binom{l}{l} c_{3-l}(X)(2H)^{1+l} = 4, \\
e_4(\mathbb{P}^4) &= e(\mathbb{P}^4) = 5.
\end{aligned}$$

On the other hand, since

$$b_i(\mathbb{P}^4) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned}
b_0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 16, \\
b_1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 10, \\
b_2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 6, \\
b_3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 0, \\
b_4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 1.
\end{aligned}$$

Example 3.4 The case where (X, L) is $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
Set $H = \mathcal{O}_{\mathbb{Q}^3}(1)$. Then $c_1(\mathbb{Q}^3) = 3H$, $c_2(\mathbb{Q}^3) = 10H^2$, $c_3(\mathbb{Q}^3) = 2H^3 = 4$.

Hence

$$\begin{aligned} e_0(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= (2H)^3 = 16, \\ e_1(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= \sum_{l=0}^{1} (-1)^l \binom{1+l}{l} c_{1-l}(X)(2H)^{2+l} = -8, \\ e_2(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= \sum_{l=0}^{2} (-1)^l \binom{l}{l} c_{2-l}(X)(2H)^{1+l} = 8, \\ e_3(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= e(\mathbb{Q}^3) = 4. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{Q}^3) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 16, \\ b_1(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 10, \\ b_2(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 6, \\ b_3(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 0. \end{aligned}$$

Example 3.5 The case where (X, L) is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
Set $H = \mathcal{O}_{\mathbb{P}^3}(1)$. Then $c_1(\mathbb{P}^3) = 4H$, $c_2(\mathbb{P}^3) = 6H^2$, $c_3(\mathbb{P}^3) = 4H^3$.

Hence

$$\begin{aligned} e_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= (3H)^3 = 27, \\ e_1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= \sum_{l=0}^{1} (-1)^l \binom{1+l}{l} c_{1-l}(X)(3H)^{2+l} = -18, \\ e_2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= \sum_{l=0}^{2} (-1)^l c_{2-l}(X)(3H)^{1+l} = 9, \\ e_3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= e(\mathbb{P}^3) = 4. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{P}^3) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 16, \\ b_1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 10, \\ b_2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 6, \\ b_3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 0. \end{aligned}$$

Example 3.6 The case where (X, L) is a Veronese fibration over a smooth curve C (see [2, (13.10)]).
Then there exists a vector bundle $\mathcal{E}$ of rank three on C such that $X = \mathbb{P}_C(\mathcal{E})$ and $L = 2H(\mathcal{E}) + f^*(B)$, where $f : X \to C$ is its fibration and $B \in \mathrm{Pic}(C)$. Set $e := \deg \mathcal{E}$ and $b := \deg B$. First we

calculate $e_i(X,L)$. Here we note that $2g(C)-2+e+2b=0$, $L^3=8e+12b$ and $g_1(X,L)=1+2e+2b$. Then

$$e_0(X,L)=L^3=8e+12b, e_1(X,L)=2-2g_1(X,L)=-4e-4b.$$

Next we calculate $e_2(X,L)$. Since

$$\begin{aligned} c_2(X) &= \sum_{j=0}^{2}\sum_{k=0}^{j}\binom{3-k}{j-k}c_k(f^*(\mathcal{E}^\vee))H(\mathcal{E})^{j-k}c_{j-k}(f^*(\mathcal{T}_C)) \\ &= 3c_1(f^*(\mathcal{T}_C))H(\mathcal{E})+3H(\mathcal{E})^2+2c_1(f^*(\mathcal{E}^\vee))H(\mathcal{E}), \end{aligned}$$

we have

$$\begin{aligned} e_2(X,L) &= \sum_{l=0}^{2}(-1)^l\binom{2+l}{l}c_{2-l}(X)(2H+f^*(B))^{1+l} \\ &= 20e+27b. \end{aligned}$$

Next we calculate $e_3(X,L)$. We note that $e_3(X,L)=e(X)$. Since

$$\begin{aligned} b_0(X) &= 1, \\ b_1(X) &= 2g(C), \\ b_2(X) &= 2, \\ b_3(X) &= 2g(C), \end{aligned}$$

we have $e_3(X,L)=e(X)=6-6g(C)=3e+6b$.

Furthermore we calculate $b_i(X,L)$. Then

$$\begin{aligned} b_0(X,L) &= 8e+12e, \\ b_1(X,L) &= 2(1+2e+2b), \\ b_2(X,L) &= 19e+25b, \\ b_3(X,L) &= 2-e-2b. \end{aligned}$$

Example 3.7 The case where (X,L) is a Del Pezzo manifold.
Here we note that by [2, (8.11) Theorem], we have $L^n\leq 8$ and (X,L) is one of the following:

(3.7.1) $(X,L)\cong(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$.

First we calculate $e_i(X,L)$. Since

$$\begin{aligned} e_i(X,L) &= \sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}c_{i-l}(X)L^{n-i+l} \\ &= \sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}\binom{n+1}{i-l}2^{n-i+l}, \end{aligned}$$

we have

$$\begin{aligned} e_0(X,L) &= \left((-1)^0\binom{2}{0}\binom{4}{0}2^0\right)2^3=8, \\ e_1(X,L) &= \left((-1)^0\binom{1}{0}\binom{4}{1}2^0+(-1)^1\binom{2}{1}\binom{4}{0}2^1\right)2^2=0, \\ e_2(X,L) &= \left((-1)^0\binom{0}{0}\binom{4}{2}2^0+(-1)^1\binom{1}{1}\binom{4}{1}2^1+(-1)^2\binom{2}{2}\binom{4}{0}2^2\right)2=4, \\ e_3(X,L) &= \left((-1)^0\binom{-1}{0}\binom{4}{3}2^0+(-1)^1\binom{0}{1}\binom{4}{2}2^1+(-1)^2\binom{1}{2}\binom{4}{1}2^2+(-1)^3\binom{2}{3}\binom{4}{0}2^3\right)2^0=4. \end{aligned}$$

Next we calculate $b_i(X,L)$. Since

$$b_j(X,\mathbb{C}) = \begin{cases} 1, & \text{if } j = 0,\ 2, \\ 0, & \text{if } j = 1,\ 3, \end{cases}$$

we have

$$\begin{aligned} b_0(X,L) &= e_0(X,L) = 8, \\ b_1(X,L) &= -e_1(X,L) + 2b_0(X) = 2, \\ b_2(X,L) &= e_2(X,L) - 2(b_0(X) - b_1(X)) = 2, \\ b_3(X,L) &= -e_3(X,L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0. \end{aligned}$$

(3.7.2) X is the blowing up of $\mathbb{P}^3$ at a point and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - E$, where $\pi : X \to \mathbb{P}^3$ is its birational morphism and E is the exceptional divisor. Then by [3, Theorem 3.2] and (3.7.1) above, we have

$$\begin{aligned} e_0(X,L) &= 7, \\ e_1(X,L) &= 0, \\ e_2(X,L) &= 5, \\ e_3(X,L) &= 6. \end{aligned}$$

and

$$\begin{aligned} b_0(X,L) &= 7, \\ b_1(X,L) &= 2, \\ b_2(X,L) &= 3, \\ b_3(X,L) &= 0. \end{aligned}$$

(3.7.3) (X,L) is either

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^*\mathcal{O}_{\mathbb{P}^1}(1)), (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^*\mathcal{O}_{\mathbb{P}^2}(1)) \text{ or } (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$$

where p_i is the ith projection and $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

(3.7.3.1) The case where $(X,L) \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^*\mathcal{O}_{\mathbb{P}^1}(1))$.
Since $\mathcal{T}_X \cong \oplus_{j=1}^3 p_j^*(\mathcal{T}_{\mathbb{P}^1})$, we have

$$\begin{aligned} c_1(\mathcal{T}_X) &= \sum_{j=1}^3 p_j^* c_1(\mathcal{T}_{\mathbb{P}^1}) \\ &= \sum_{j=1}^3 p_j^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)), \\ c_2(\mathcal{T}_X) &= p_1^*c_1(\mathcal{T}_{\mathbb{P}^1})p_2^*c_1(\mathcal{T}_{\mathbb{P}^1}) + p_1^*c_1(\mathcal{T}_{\mathbb{P}^1})p_3^*c_1(\mathcal{T}_{\mathbb{P}^1}) + p_2^*c_1(\mathcal{T}_{\mathbb{P}^1})p_3^*c_1(\mathcal{T}_{\mathbb{P}^1}) \\ &= p_1^*c_1(\mathcal{O}_{\mathbb{P}^1}(2))p_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(2)) + p_1^*c_1(\mathcal{O}_{\mathbb{P}^1}(2))p_3^*c_1(\mathcal{O}_{\mathbb{P}^1}(2)) + p_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(2))p_3^*c_1(\mathcal{O}_{\mathbb{P}^1}(2)) \\ c_3(X) &= e(X). \end{aligned}$$

On the other hand

$$\begin{aligned} b_0(X) &= 1, \\ b_1(X) &= 0, \\ b_2(X) &= 3, \\ b_3(X) &= 0. \end{aligned}$$

Therefore

$$\begin{aligned}
e_0(X,L) &= L^3 = 6,\\
e_1(X,L) &= \sum_{l=0}^{1}(-1)^l\binom{1+l}{l}c_{1-l}(X)L^{2+l} = 0,\\
e_2(X,L) &= \sum_{l=0}^{2}(-1)^l\binom{l}{l}c_{2-l}(X)L^{1+l} = 6,\\
e_3(X,L) &= e(X) = 8,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X,L) &= e_0(X,L) = 6,\\
b_1(X,L) &= -e_1(X,L) + 2b_0(X) = 2,\\
b_2(X,L) &= e_2(X,L) - 2(b_0(X) - b_1(X)) = 4,\\
b_3(X,L) &= -e_3(X,L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0.
\end{aligned}$$

(3.7.3.2) The case where $(X,L) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^{2} p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$.
Since $\mathcal{T}_X \cong \oplus_{j=1}^{2} p_j^*(\mathcal{T}_{\mathbb{P}^2})$, we have

$$\begin{aligned}
c_1(\mathcal{T}_X) &= \sum_{j=1}^{2} p_j^* c_1(\mathcal{T}_{\mathbb{P}^2})\\
&= \sum_{j=1}^{2} p_j^* c_1(\mathcal{O}_{\mathbb{P}^2}(3)),\\
c_2(\mathcal{T}_X) &= p_1^*c_2(\mathcal{T}_{\mathbb{P}^2}) + p_1^*c_1(\mathcal{T}_{\mathbb{P}^2})p_2^*c_1(\mathcal{T}_{\mathbb{P}^2}) + p_2^*c_2(\mathcal{T}_{\mathbb{P}^2})\\
&= 3p_1^*\mathcal{O}_{\mathbb{P}^2}(1)^2 + 9p_1^*\mathcal{O}_{\mathbb{P}^2}(1)p_2^*\mathcal{O}_{\mathbb{P}^2}(1) + 3p_2^*\mathcal{O}_{\mathbb{P}^2}(1)^2,\\
c_3(\mathcal{T}_X) &= p_1^*c_2(\mathcal{T}_{\mathbb{P}^2})p_2^*c_1(\mathcal{T}_{\mathbb{P}^2}) + p_1^*c_1(\mathcal{T}_{\mathbb{P}^2})p_2^*c_2(\mathcal{T}_{\mathbb{P}^2})\\
&= 9p_1^*\mathcal{O}_{\mathbb{P}^2}(1)^2p_2^*\mathcal{O}_{\mathbb{P}^2}(1) + 9p_1^*\mathcal{O}_{\mathbb{P}^2}(1)p_2^*\mathcal{O}_{\mathbb{P}^2}(1)^2,\\
c_4(X) &= e(X).
\end{aligned}$$

On the other hand

$$\begin{aligned}
b_0(X) &= 1,\\
b_1(X) &= 2q(X) = 0,\\
b_2(X) &= 2b_2(\mathbb{P}^2)b_0(\mathbb{P}^2) + b_1(\mathbb{P}^2)b_1(\mathbb{P}^2) = 2,\\
b_3(X) &= 2b_3(\mathbb{P}^2)b_0(\mathbb{P}^2) + 2b_2(\mathbb{P}^2)b_1(\mathbb{P}^2) = 0,\\
b_4(X) &= 2b_4(\mathbb{P}^2)b_0(\mathbb{P}^2) + 2b_3(\mathbb{P}^2)b_1(\mathbb{P}^2) + b_2(\mathbb{P}^2)b_2(\mathbb{P}^2) = 3.
\end{aligned}$$

Therefore

$$\begin{aligned}
e_0(X,L) &= L^4 = 6,\\
e_1(X,L) &= \sum_{l=0}^{1}(-1)^l\binom{2+l}{l}c_{1-l}(X)L^{3+l} = 0,\\
e_2(X,L) &= \sum_{l=0}^{2}(-1)^l\binom{1+l}{l}c_{2-l}(X)L^{2+l} = 6,\\
e_3(X,L) &= \sum_{l=0}^{3}(-1)^l\binom{l}{l}c_{3-l}(X)L^{1+l} = 6,\\
e_4(X,L) &= e(X) = 9,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X,L) &= e_0(X,L)=6,\\
b_1(X,L) &= -e_1(X,L)+2b_0(X)=2,\\
b_2(X,L) &= e_2(X,L)-2(b_0(X)-b_1(X))=4,\\
b_3(X,L) &= -e_3(X,L)+2(b_0(X)-b_1(X)+b_2(X))=0,\\
b_4(X,L) &= e_4(X,L)-2(b_0(X)-b_1(X)+b_2(X)-b_3(X))=3.
\end{aligned}$$

(3.7.3.3) The case where $(X,L)\cong(\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}),H(T_{\mathbb{P}^2}))$.
First we note that

$$\begin{aligned}
b_0(X) &= 1,\\
b_1(X) &= 0,\\
b_2(X) &= 2,\\
b_3(X) &= 0.
\end{aligned}$$

Then by [4, Corollary 3.1 (3.1.2) and Corollary 3.3 (3.3.2)] we have

$$\begin{aligned}
e_0(X,L) &= s_2(\mathcal{T}_{\mathbb{P}^2})=K_{\mathbb{P}^2}^2-c_2(\mathbb{P}^2)=6,\\
e_1(X,L) &= -(c_1(\mathcal{T}_{\mathbb{P}^2})+K_{\mathbb{P}^2})c_1(\mathcal{T}_{\mathbb{P}^2})=0,\\
e_2(X,L) &= c_2(\mathbb{P}^2)+c_2(\mathcal{T}_{\mathbb{P}^2})=6,\\
e_3(X,L) &= 2e(\mathbb{P}^2)=6,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X,L) &= e_0(X,L)=6,\\
b_1(X,L) &= (c_1(\mathcal{T}_{\mathbb{P}^2})+K_{\mathbb{P}^2})c_1(\mathcal{T}_{\mathbb{P}^2})+2=2,\\
b_2(X,L) &= b_2(X)+c_2(\mathbb{P}^2)-1=4,\\
b_3(X,L) &= b_3(X)=0.
\end{aligned}$$

(3.7.4) The case where (X,L) is a linear section of the Grassmann variety $\mathrm{Gr}(5,2)$ parametrizing lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ via the Plücker embedding. Then $L^n=5$.

First we review the Chern class of $\mathrm{Gr}(p,q)$ parametrizing $\mathbb{P}^{q-1}$ in $\mathbb{P}^{p-1}$. Let S (resp. Q) be the universal subbundle (resp. the universal quotient bundle) of $\mathrm{Gr}(p,q)$. Then

$$c(\mathrm{Gr}(p,q))=c(S^\vee\otimes Q). \tag{1}$$

We note that $\mathrm{rank}S=q$ and $\mathrm{rank}Q=p-q$. From (1),

$$ch(\mathcal{T}_{\mathrm{Gr}(p,q)})=ch(S^\vee)ch(Q) \tag{2}$$

holds. Since $ch(Q)+ch(S)=p$, we have

$$ch(S)=q-\sum_{k\geq 1}ch_k(Q).$$

On the other hand

$$ch_k(S^\vee)=q-\sum_{k\geq 1}(-1)^k ch_k(Q). \tag{3}$$

Next we explain the Schubert caliculas. For $\lambda=(\lambda_0,\dots,\lambda_d)$ with $p-q\geq\lambda_0\geq\dots\geq\lambda_d\geq 0$, we set

$$\{\lambda_0,\dots,\lambda_d\}=\det(c_{\lambda_{i+j-1}}(Q))_{0\leq i,j\leq p}.$$

Then $c_m(Q) = \{m, 0, \ldots, 0\}$. We note that the following equality holds.

$$\{\lambda\} \cdot c_m(Q) = \sum \{\mu\}, \tag{4}$$

where the sum over μ with $p - q \geq \mu_0 \geq \lambda_0 \geq \cdots \geq \mu_p \geq \lambda_p$ and $\sum_{i=0}^{p} \lambda_i = m + \sum_{i=0}^{p} \mu_i$. Moreover we have

$$\int c_1(Q)^s \{\lambda_0, \ldots, \lambda_d\} = \frac{k!}{a_0! \cdots a_d!} \prod_{i<j} (a_j - a_i). \tag{5}$$

Here $a_i = p - q + i - \lambda_i$, $k = \sum_{i=0}^{p} a_i - \frac{p(p+1)}{2}$ and $s = \dim \mathrm{Gr}(p, q) - \sum_{i=0}^{d} (\lambda_i - i) = q(p - q) - \sum_{i=0}^{d} (\lambda_i - i)$.

Here we consider thde case where $p = 5$ and $q = 2$. Then first we calculate $c_j(\mathrm{Gr}(5, 2))$ for $1 \leq j \leq 5$. From (2) and (3), we have

$$\begin{aligned} ch(\mathrm{Gr}(5,2)) &= ch(S^\vee) ch(Q) \\ &= \left(2 - \sum_{k \geq 1} (-1)^k ch_k(Q)\right) \left(3 + \sum_{k \geq 1} ch_k(Q)\right). \end{aligned}$$

Using this, we get the following. (Here we note that $c_j(Q) = 0$ for $j \geq 4$ because rank$Q = 3$.)

$$\begin{aligned} c_1(\mathrm{Gr}(5,2)) &= 5c_1(Q) \\ c_2(\mathrm{Gr}(5,2)) &= 12c_1(Q)^2 - c_2(Q) \\ c_3(\mathrm{Gr}(5,2)) &= 20c_1(Q)^3 - 10c_1(Q)c_2(Q) + 5c_3(Q) \\ c_4(\mathrm{Gr}(5,2)) &= 28c_1(Q)^4 - 38c_1(Q)^2 c_2(Q) + 20c_1(Q)c_3(Q) + 7c_2(Q)^2 \\ c_5(\mathrm{Gr}(5,2)) &= 36c_1(Q)^5 - 90c_1(Q)^3 c_2(Q) + 40c_1(Q)^2 c_3(Q) + 45c_1(Q)c_2(Q)^2 - 10c_2(Q)c_3(Q). \end{aligned}$$

Next we use the Scubert caliculas. First from (5) we get the following.

$$\begin{aligned} c_1(Q)^6 &= 5, \\ c_1(Q)^4 c_2(Q) &= 3, \\ c_1(Q)^3 c_3(Q) &= 1. \end{aligned}$$

Next we calculate $c_2(Q)^2 c_1(Q)^2$. Since $\{2, 0\} \cdot \{2, 0\} = \{3, 1\} + \{2, 2\}$, we have

$$\begin{aligned} c_2(Q)^2 c_1(Q)^2 &= \int c_1(Q)^3 \{3, 1\} + \int c_1(Q)^3 \{2, 2\} \\ &= 2. \end{aligned}$$

Next we calculate $c_2(Q)c_3(Q)c_1(Q)$. Since $\{2, 0\} \cdot \{3, 0\} = \{3, 2\}$, we have

$$\begin{aligned} c_2(Q)c_3(Q)c_1(Q) &= \int c_1(Q)^2 \{3, 2\} \\ &= 1. \end{aligned}$$

Hence

$$\begin{aligned} c_1(\mathrm{Gr}(5,2)) L^5 &= 5c_1(Q)^6 = 25 \\ c_2(\mathrm{Gr}(5,2)) L^4 &= 12c_1(Q)^6 - c_1(Q)^{64} c_2(Q) = 57 \\ c_3(\mathrm{Gr}(5,2)) L^3 &= 20c_1(Q)^{36} - 10c_1(Q)^4 c_2(Q) + 5c_1(Q)^3 c_3(Q) = 75 \\ c_4(\mathrm{Gr}(5,2)) L^2 &= 28c_1(Q)^6 - 38c_1(Q)^4 c_2(Q) + 20c_1(Q)^3 c_3(Q) + 7c_1(Q)^2 c_2(Q)^2 = 60 \\ c_5(\mathrm{Gr}(5,2)) L &= 36c_1(Q)^6 - 90c_1(Q)^4 c_2(Q) + 40c_1(Q)^3 c_3(Q) \\ &\quad + 45c_1(Q)^2 c_2(Q)^2 - 10c_1(Q)c_2(Q)c_3(Q) \\ &= 30. \end{aligned}$$

Therefore

$$
\begin{aligned}
e_0(X,L) &= L^6 = 5,\\
e_1(X,L) &= c_1(X)L^5 - 5L^6 = 0,\\
e_2(X,L) &= c_2(X)L^4 - 4c_1(X)L^5 + 10L^6 = 7,\\
e_3(X,L) &= c_3(X)L^3 - 3c_2(X)L^4 + 6c_1(X)L^5 - 10L^6 = 4,\\
e_4(X,L) &= c_4(X)L^2 - 2c_3(X)L^3 + 3c_2(X)L^4 - 4c_1(X)L^5 + 5L^6 = 6,\\
e_5(X,L) &= c_5(X)L - c_4(X)L^2 + c_3(X)L^3 - c_2(X)L^4 + c_1(X)L^5 - L^6 = 8,\\
e_6(X,L) &= e(X) = 10.
\end{aligned}
$$

Next we calculate $b_i(X,L)$. Since $b_0(X) = b_2(X) = 1$, $b_4(X) = b_6(X) = b_8(X) = 2$ $b_{10}(X) = b_{12}(X) = 1$ and $b_j(X) = 0$ for every positive odd integer j, we have

$$
\begin{aligned}
b_0(X,L) &= 5,\\
b_1(X,L) &= 2,\\
b_2(X,L) &= 5,\\
b_3(X,L) &= 0,\\
b_4(X,L) &= 2,\\
b_5(X,L) &= 4,\\
b_6(X,L) &= 2.
\end{aligned}
$$

(3.7.5) The case where (X,L) is a complete intersection of two hyperquadrics in $\mathbb{P}^{n+2}$. Then $L^n = 4$. First we calculate $e(X)$ in this case. In general we can prove the following.

Lemma 3.1 *Let (X,L) be a complete intersection of two hypersurfaces of degree s and t in $\mathbb{P}^{n+2}$. Then*

$$
e(X) = -\frac{s}{t^2}(1-t)^{n+3}\sum_{k=0}^{n-1}\left(\frac{s}{t}\right)^k + \frac{s}{t^2}\sum_{j=0}^{n-1}\left(\frac{s}{t}\right)^j\sum_{k=0}^{2+j}(-t)^k\binom{n+3}{n+3-k} + (-s)^{n+1}(-t).
$$

Proof. Let $c_j := c_j(X)$ and $H := \mathcal{O}_X(1)$. Then the following holds (see [7, Example 3.2.12]).

$$
(1+H)^{n+3} = C(X)(1+sH)(1+tH).
$$

Here $C(X) = (1 + c_1 + \cdots + c_n)$. Hence

$$
\begin{aligned}
(c_n + sc_{n-1}H) + t(c_{n-1}H + sc_{n-2}H^2) &= \binom{n+3}{n}H^n\\
(c_{n-1} + sc_{n-2}H) + t(c_{n-2}H + sc_{n-3}H^2) &= \binom{n+3}{n-1}H^{n-1}\\
&\ \vdots\\
(c_2 + sc_1H) + t(c_1H + sc_0H^2) &= \binom{n+3}{2}H^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
&c_n + sc_{n-1}H + (-t)^{n-2}\cdot t(c_1H^{n-1} + sc_0H^n)\\
&= \left(\binom{n+3}{n} + (-t)\binom{n+3}{n-1} + \cdots + (-t)^{n-3}\binom{n+3}{3} + (-t)^{n-2}\binom{n+3}{2}\right)H^n
\end{aligned}
$$

Moreover since $c_1 H^{n-1} = \mathcal{O}(n-s-t+3)H^{n-1}$, we have $c_1 H^{n-1} + sc_0 H^n = (n-t+3)H^n$. Therefore

$$\begin{aligned}
& c_n + sc_{n-1}H \\
&= \left(\binom{n+3}{n} + (-t)\binom{n+3}{n-1} + \cdots \right. \\
&\quad \left. \cdots + (-t)^{n-3}\binom{n+3}{3} + (-t)^{n-2}\binom{n+3}{2} + (-t)^{n-1}\binom{n+3}{1} + (-t)^n \right) H^n \\
&= -\frac{s}{t^2}\left((-t)^3\binom{n+3}{n} + (-t)^4\binom{n+3}{n-1} + \cdots \right. \\
&\quad \left. \cdots + (-t)^n\binom{n+3}{3} + (-t)^{n+1}\binom{n+3}{2} + (-t)^{n+2}\binom{n+3}{1} + (-t)^{n+3} \right) \\
&= -\frac{s}{t^2}\left((1-t)^{n+3} - 1 - (-t)^1\binom{n+3}{n+2} - (-t)^2\binom{n+3}{n+1} \right) \\
&= -\frac{s}{t^2}\left((1-t)^{n+3} - 1 + t\binom{n+3}{n+2} - t^2\binom{n+3}{n+1} \right).
\end{aligned}$$

By the same argument as above for every j with $1 \le j \le n-1$ we have

$$\begin{aligned}
& c_j H^{n-j} + sc_{j-1}H^{n-j+1} \\
&= \frac{s}{(-t)^{n-j+2}}\left((1-t)^{n+3} - \sum_{k=0}^{n+2-j} (-t)^k \binom{n+3}{n+3-k} \right).
\end{aligned}$$

Hence

$$c_n = -\frac{s}{t^2}(1-t)^{n+3}\sum_{k=0}^{n-1}\left(\frac{s}{t}\right)^k + \frac{s}{t^2}\sum_{j=0}^{n-1}\left(\frac{s}{t}\right)^j \sum_{k=0}^{2+j}(-t)^k\binom{n+3}{n+3-k} + (-s)^{n+1}(-t). \qquad (6)$$

□

Lemma 3.2 *Let (X, L) be a complete intersection of two hyperquadrics in $\mathbb{P}^{n+2}$. Then*

$$e(X) = \begin{cases} 2n+4, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Lemma 3.1 we have

$$c_n = (-2)^{n+2} + \frac{1}{2}\left(n(-1)^n + \sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\binom{n+3}{n+3-k} \right).$$

Next we prove the following.

Claim 3.1

$$\begin{aligned}
& (-2)^{n+2} + \frac{1}{2}\left(n(-1)^n + \sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\binom{n+3}{n+3-k} \right) \qquad (7) \\
&= \begin{cases} 0, & n \text{ is odd,} \\ 2n+4, & n \text{ is even.} \end{cases}
\end{aligned}$$

Proof. First we note the following.

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\binom{n+2}{n+2-k} &(8)\\
&=\sum_{j=1}^{n}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+\sum_{j=1}^{n}(-2)^{n+2-j}\binom{n+2}{j}\\
&=\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+\sum_{k=0}^{1}(-2)^k\binom{n+2}{n+2-k}+\sum_{j=1}^{n}(-2)^{n+2-j}\binom{n+2}{j},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\binom{n+2}{n+3-k} &= \sum_{j=1}^{n}\sum_{k=1}^{n+2-j}(-2)^k\binom{n+2}{n+3-k} \qquad (9)\\
&= \sum_{j=1}^{n}\sum_{k=0}^{n+1-j}(-2)^{k+1}\binom{n+2}{n+2-k}\\
&= \sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^{k+1}\binom{n+2}{n+2-k}\\
&\quad+\sum_{k=0}^{1}(-2)^{k+1}\binom{n+2}{n+2-k}\\
&= -2\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}\\
&\quad+\sum_{k=0}^{1}(-2)^{k+1}\binom{n+2}{n+2-k}.
\end{aligned}
$$

Then from (8) and (9) we have

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\left(\binom{n+2}{n+2-k}+\binom{n+2}{n+2-k}\right)\\
&=-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}-\sum_{k=0}^{1}(-2)^k\binom{n+2}{n+2-k}+\sum_{j=1}^{n}(-2)^{n+2-j}\binom{n+2}{j}\\
&=-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+4n+6+(-1)^{n+2}-(-2)^{n+2}. &(10)
\end{aligned}
$$

Here we prove (7) by induction on n.

If $n=1$ and 2, then (7) holds.

Next we assume that (7) holds for $n-1$ is odd. Then by assumption we have the following equality.

$$
(-2)^{n+1}+\frac{1}{2}\left((n-1)(-1)^{n-1}+\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}\right) = 0. \qquad (11)
$$

Then by using (11), we have

$$
\begin{aligned}
&(-2)^{n+2}+\frac{1}{2}\left(n(-1)^n+\sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\binom{n+3}{n+3-k}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(n(-1)^n-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+4n+6\right.\\
&\qquad\left.+(-1)^{n+2}-(-2)^{n+2}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(n-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+4n+6+(-1)^{n+2}-(-2)^{n+2}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(n-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+4n+6+1-(-2)^{n+2}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(5n+7+2(-2)^{n+1}(n-1)(-1)^{n-1}\right)\\
&=2n+4.
\end{aligned}
$$

Next we assume that (7) holds for $n-1$ is even. Then by assumption we have the following equality.

$$
(-2)^{n+1}+\frac{1}{2}\left((n-1)(-1)^{n-1}+\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}\right) \quad = \quad 2n+2. \tag{12}
$$

Then by using (12), we have

$$
\begin{aligned}
&(-2)^{n+2}+\frac{1}{2}\left(n(-1)^n+\sum_{j=1}^{n}\sum_{k=0}^{n+2-j}(-2)^k\binom{n+3}{n+3-k}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(-n-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}+4n+6+(-1)^{n+2}-(-2)^{n+2}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(3n+5-\sum_{j=1}^{n-1}\sum_{k=0}^{n+1-j}(-2)^k\binom{n+2}{n+2-k}-(-2)^{n+2}\right)\\
&=(-2)^{n+2}+\frac{1}{2}\left(3n+5+2(-2)^{n+1}+(n-1)(-1)^{n-1}-2(2n+2)-(-2)^{n+2}\right)\\
&=0.
\end{aligned}
$$

This completes the proof of Claim 3.1. □

From Claim 3.1 we get Lemma 3.2. □

Remark 3.1 Let (X, L) be a complete intersection of two hypersurfaces of degree s and t in $\mathbb{P}^{n+2}$. Then from (6) we can write $e(X)$ as follows.

$$
e(X)=(-1)^n st\left(\sum_{k=0}^{n}(-1)^k\binom{n+3}{k}\left(\sum_{j=0}^{n-k}s^{n-k-j}t^j\right)\right).
$$

Proof.

$$\begin{aligned}
c_n &= -\frac{s}{t^2}(1-t)^{n+3}\left(1+\frac{s}{t}+\cdots+\left(\frac{s}{t}\right)^{n-1}\right)+\frac{s}{t^2}\left\{\left(1+(-t)\binom{n+3}{n+2}+(-t)^2\binom{n+3}{n+1}\right)\right. \\
&\quad +\frac{s}{t}\left(1+(-t)\binom{n+3}{n+2}+(-t)^2\binom{n+3}{n+1}+(-t)^3\binom{n+3}{n}\right) \\
&\quad \left.+\cdots+\left(\frac{s}{t}\right)^{n-1}\left(1+(-t)\binom{n+3}{n+2}+\cdots+(-t)^{n+1}\binom{n+3}{2}\right)\right\}+(-s)^{n+1}(-t) \\
&= -\frac{s}{t^2}\left((-t)^3\binom{n+3}{n}+\cdots+(-t)^{n+3}\right)\left(1+\frac{s}{t}+\cdots+\left(\frac{s}{t}\right)^{n-1}\right) \\
&\quad +\frac{s}{t^2}\left(\sum_{j=1}^{n-1}\left(\frac{s}{t}\right)^j\sum_{k=P1}^{j}(-t)^{2+k}\binom{n+3}{n+1-k}\right)+(-s)^{n+1}(-t) \\
&= -\frac{s}{t^2}\left((-t)^3\binom{n+3}{n}+\cdots+(-t)^{n+3}\right)-\frac{s^2}{t^3}\left((-t)^4\binom{n+3}{n-1}+\cdots+(-t)^{n+3}\right) \\
&\quad -\frac{s^3}{t^4}\left((-t)^5\binom{n+3}{n-2}+\cdots+(-t)^{n+3}\right)\cdots-\frac{s^n}{t^{n+1}}\left((-t)^{n+2}\binom{n+3}{1}+(-t)^{n+3}\right) \\
&\quad +(-s)^{n+1}(-t) \\
&= (-s)\left((-t)\binom{n+3}{n}+\cdots+(-t)^{n+1}\right)+s^2\left((-t)\binom{n+3}{n-1}+\cdots+(-t)^n\right) \\
&\quad -s^3\left((-t)\binom{n+3}{n-2}+\cdots+(-t)^{n-1}\right)\cdots+(-s)^n\left((-t)\binom{n+3}{1}+(-t)^2\right)+(-s)^{n+1}(-t) \\
&= \sum_{j=1}^{n+1}(-s)(-t)^j\binom{n+3}{n+1-j}+\sum_{j=1}^{n}(-s)^2(-t)^j\binom{n+3}{n-j}+\sum_{j=1}^{n-1}(-s)^3(-t)^j\binom{n+3}{n-1-j} \\
&\quad +\cdots+\sum_{j=1}^{2}(-s)^n(-t)^j\binom{n+3}{2-j}+\sum_{j=1}^{1}(-s)^{n+1}(-t)^j\binom{n+3}{1-j} \\
&= st\left(\sum_{j=1}^{n+1}(-t)^{j-1}\binom{n+3}{n+1-j}+\sum_{j=1}^{n}(-s)(-t)^{j-1}\binom{n+3}{n-j}+\cdots+\sum_{j=1}^{1}(-s)^n(-t)^{j-1}\binom{n+3}{1-j}\right) \\
&= st\left(\sum_{j=0}^{n}(-t)^j\binom{n+3}{n-j}+\sum_{j=0}^{n-1}(-s)(-t)^j\binom{n+3}{n-1-j}+\cdots+\sum_{j=0}^{0}(-s)^n(-t)^j\binom{n+3}{-j}\right) \\
&= st\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n+3}{k}\left(\sum_{j=0}^{n-k}s^{n-k-j}t^j\right)\right) \\
&= (-1)^n st\left(\sum_{k=0}^{n}(-1)^k\binom{n+3}{k}\left(\sum_{j=0}^{n-k}s^{n-k-j}t^j\right)\right).
\end{aligned}$$

So we get the assertion. □

Here we go back to the case (3.7.5). In this case, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L such that (X_j, L_j) is complete intersection of two hyperquadrics in $\mathbb{P}^{n-j+2}$. Since $e_i(X,L) = e(X_{n-i})$, we see that

$$e_i(X,L) = \begin{cases} 2i+4, & \text{if } i \text{ is even with } i \geq 2, \\ 0, & \text{if } i \text{ is odd with } i \geq 3. \end{cases}$$

We also note that

$$e_i(X,L) = \begin{cases} 4, & \text{if } i = 0, \\ 0, & \text{if } i = 1. \end{cases}$$

Next we calculate $b_i(X,L)$. Since

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-1, \\ 0, & \text{if } i \text{ is odd with } i \leq n-1, \end{cases}$$

we have

$$b_i(X,L) = \begin{cases} 2i+4-2\frac{i}{2} = i+4, & \text{if } i \text{ is even with } i \geq 2, \\ 0+2\frac{i+1}{2} = i+1, & \text{if } i \text{ is odd with } i \geq 3. \end{cases}$$

We also note that

$$b_i(X,L) = \begin{cases} 4, & \text{if } i = 0, \\ 2, & \text{if } i = 1. \end{cases}$$

(3.7.6) The case where X is a hypercubic in $\mathbb{P}^{n+1}$ and $L = \mathcal{O}_X(1)$.

Here we consider more general case than this. In general we can prove the following claim.

Lemma 3.3 *Let (X,L) be a polarized manifold of dimension n such that X is a hypersurface of degree m and $L = \mathcal{O}_X(1)$. Then*

$$\begin{aligned} e_i(X,L) &= \frac{1}{m}\left((1-m)^{i+2} - 1 + m(i+2)\right), \\ b_i(X,L) &= \begin{cases} \frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) - i, & \textit{if } i \textit{ is even with } i \leq n-1, \\ -\frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) + i + 1, & \textit{if } i \textit{ is odd with } i \leq n-1. \end{cases} \end{aligned}$$

Proof. First we calculate $e_n(X,L)$. Let $c_j := c_j(X)$ and $H := \mathcal{O}_X(1)$. Then the following holds (see [7, Example 3.2.12]).

$$(1+H)^{n+2} = (1+c_1+\cdots+c_n)(1+mH).$$

Hence

$$\begin{aligned} c_n + mc_{n-1}H &= \binom{n+2}{n}H^n \\ c_{n-1} + mc_{n-2}H &= \binom{n+2}{n-1}H^{n-1} \\ &\vdots \\ c_1 + mH &= \binom{n+2}{1}H \end{aligned}$$

So we have

$$\begin{aligned} c_n &= (-m)^n H^n + \frac{1}{m^2}\left((-m)^2\binom{n+2}{2} + (-m)^3\binom{n+2}{3} + \cdots + (-m)^{n+1}\binom{n+2}{n+1}\right)H^n \\ &= m(-m)^n + \frac{1}{m^2}\left((1-m)^{n+2} - 1 - (-m)\binom{n+2}{1} - (-m)^{n+2}\right)m \\ &= \frac{1}{m}\left((1-m)^{n+2} - 1 + m(n+2)\right). \end{aligned}$$

On the other hand, in this case, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L such that (X_j, L_j) is a hypersurface of degree m in $\mathbb{P}^{n-j+1}$. Since $e_i(X,L) = e(X_{n-i})$, by the above argument we see that

$$e_i(X,L) = \frac{1}{m}\left((1-m)^{i+2} - 1 + m(i+2)\right).$$

Next we calculate $b_i(X,L)$. Since

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-1, \\ 0, & \text{if } i \text{ is odd with } i \leq n-1, \end{cases}$$

we have

$$\begin{aligned} b_i(X,L) &= \begin{cases} \frac{1}{m}\left((1-m)^{i+2} - 1 + m(i+2)\right) - 2 \cdot \frac{i}{2}, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m}\left((1-m)^{i+2} - 1 + m(i+2)\right) + 2 \cdot \frac{i+1}{2}, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases} \\ &= \begin{cases} \frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) - i, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) + i + 1, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases} \end{aligned}$$

This completes the proof of Lemma 3.3. □

(3.7.7) The case where X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 4, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$. Here we consider more general case than this.

Lemma 3.4 *Let X be a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree m with even $m \geq 4$, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$. Then*

$$\begin{aligned} e_i(X,L) &= i + 2 - \frac{1}{m}(m - 1 + (1-m)^{i+1}), \\ b_i(X,L) &= \left(i + 2 - \frac{1}{m}(m - 1 + (1-m)^{i+1})\right) + (-1)^{i+1}\begin{cases} i & \textit{if } i \textit{ is even,} \\ i+1 & \textit{if } i \textit{ is odd.} \end{cases} \end{aligned}$$

Proof. First we calculate $e_n(X,L)$. Let B be the branch locus. Then

$$e(X) = 2e(\mathbb{P}^n) - e(B).$$

Hence by Lemma 3.3

$$\begin{aligned} e_n(X,L) &= e(X) \\ &= 2e(\mathbb{P}^n) - e(B) \\ &= 2n + 2 - \frac{1}{m}\left((1-m)^{n+1} + m(n+1) - 1\right) \\ &= n + 2 - \frac{1}{m}(m - 1 + (1-m)^{n+1}). \end{aligned}$$

Next we consider $e_i(X,L)$. First we note that $\Delta(X,L) = 1$ in this case. Since $\mathrm{Bs}|L| = \emptyset$, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L. Then we see that $\Delta(X_j, L_j) = 1$ and $L_j^{n-j} = 2$, where $L_j := L|_{X_j}$ because $g(X,L) = m/2 - 1 \geq 1 = \Delta(X,L)$ and $L^n = 2 = 2\Delta(X,L)$. Hence X_j is a double covering of $\mathbb{P}^{n-j}$ branched along a smooth hypersurface of degree 4, and L_j is the pull-back of $\mathcal{O}_{\mathbb{P}^{n-j}}(1)$. Since $e_i(X,L) = e(X_{n-i})$, by the above argument we see that for every integer i with $i \geq 1$, we have

$$e_i(X,L) = i + 2 - \frac{1}{m}(m - 1 + (1-m)^{i+1}). \qquad (13)$$

Here we note that $e_0(X,L) = L^n = 2 = 0 + 2 - \frac{1}{4}(3 + (-3)^{0+1})$. Hence (13) also holds for $i = 0$.

Next we calculate $b_i(X,L)$. By the Barth-type theorem (see e.g. [8, Theorem 7.1.15]), we have

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-1, \\ 0, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases}$$

Hence we have

$$\begin{aligned} b_i(X,L) &= (-1)^i \left(e_i(X,L) - 2\sum_{j=0}^{i-1}(-1)^j b_j(X) \right) \\ &= (-1)^i \left(i + 2 - \frac{1}{m}(m - 1 + (1-m)^{i+1}) \right) - 2(-1)^i \cdot \begin{cases} \frac{i-1+1}{2} & \text{if } i \text{ is even,} \\ \frac{i-1}{2} + 1 & \text{if } i \text{ is odd,} \end{cases} \\ &= \left(i + 2 - \frac{1}{m}(m - 1 + (1-m)^{i+1}) \right) + (-1)^{i+1} \begin{cases} i & \text{if } i \text{ is even,} \\ i+1 & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

We get the assertion of Lemma 3.4. □

(3.7.8) The case where (X,L) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3,2,1,\dots,1)$. Then $L^n = 1$ and $\mathrm{Bs}|L| = \{p\}$ (see [1, (16.7) Theorem and Appendix 1]).

In this case, there exists a smooth ladder $X \supset X_1 \supset \cdots \supset X_{n-1}$ of L such that (X_j, L_j) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3,2,1,\dots,1)$. Since $e_i(X,L) = e(X_{n-i})$, in order to calculate $e_i(X,L)$ for $i \geq 1$, it suffices to calculate $e(X)$.

Let $\pi : X^* \to X$ be the blowing up at $p \in X$. Then $\pi^*(L) - E$ is base point free and let $f : X^* \to \mathbb{P}^{n-1}$ be the morphism defined by $|\pi^*(L) - E|$. In this case, there exists a projective bundle $p : V \to \mathbb{P}^{n-1}$ and a double covering $\rho : X^* \to V$ such that $f = p \circ \rho$. Here we note that $V = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$. Let H_V be the tautological line bundle of V and let B be the branch locus of ρ. Then there exist $B_1 \in |H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)|$ and $B_2 \in |3H_V|$ such that $B_1 \cong \mathbb{P}^{n-1}$ and $B = B_1 + B_2$. Here we note that the following equality holds.

$$\begin{aligned} e(X) &= e(X^*) - e(E) + 1, & (14) \\ e(X^*) &= 2e(V) - e(B), & (15) \\ e(B) &= e(B_1) + e(B_2). & (16) \end{aligned}$$

Therefore in order to calculate $e(X)$, we need the value of $e(E)$, $e(B_1)$, $e(B_2)$, and $e(V)$.

First we note that

$$e(E) = e(\mathbb{P}^{n-1}) = n \tag{17}$$

and

$$e(B_1) = e(\mathbb{P}^{n-1}) = n. \tag{18}$$

Next we calculate $e(V)$. By [1, Proof of Lemma in Appendix 2], we see that there exist the following three exact sequence:

$$0 \to 2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) \to \mathcal{T}_V \to \mathcal{T}_{\mathbb{P}^{n-1}}|_V \to 0, \tag{19}$$

$$0 \to \mathcal{O}_V \to H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))^\vee \otimes \pi^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \to \mathcal{T}_{\mathbb{P}^{n-1}}|_V \to 0, \tag{20}$$

$$0 \to \mathcal{T}_{B_2} \to \mathcal{T}_V|_{B_2} \to (3H_V)|_{B_2} \to 0. \tag{21}$$

From (19), we have

$$c(\mathcal{T}_V) = c(2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c(\mathcal{T}_{\mathbb{P}^{n-1}}|_V). \tag{22}$$

Hence

$$\begin{aligned} c_n(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c_{n-1}(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) \\ &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))(n(\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}). \end{aligned}$$

By (20), we get

$$c((\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{\oplus n}) = c(\mathcal{O}_V)c(\pi^*\mathcal{T}_{\mathbb{P}^{n-1}}). \tag{23}$$

Hence

$$c_{n-1}(p^*\mathcal{T}_{\mathbb{P}^{n-1}}) = \binom{n}{n-1}\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}.$$

Therefore

$$\begin{aligned} c_n(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))(n(\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}) \\ &= 2nH_V\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1} \\ &= 2n. \end{aligned} \tag{24}$$

Next we calculate $e(B_2)$. Before this, we note the following. Let $\mathcal{E} := \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}$ and let $H(\mathcal{E})$ be the tautological line bundle of $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$. Then $V = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$ and $H(\mathcal{E}) = H_V$. In this case, since $c_j(\mathcal{E}) = 0$ for any $j \geq 2$, we have $s_j(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^{n-1}}(2)^j$. Therefore

$$H_V^j\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-j} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-j}s_{j-1}(\mathcal{E}) = 2^{j-1}. \tag{25}$$

From (21), we have

$$c(\mathcal{T}|_{B_2}) = c(\mathcal{T}_{B_2})c(3H_V|_{B_2}). \tag{26}$$

From (26) we obtain the following:

$$\begin{aligned} c_{n-1}(B_2) + c_{n-2}(B_2)(3H_V|_{B_2}) &= c_{n-1}(V)B_2 \\ c_{n-2}(B_2) + c_{n-3}(B_2)(3H_V|_{B_2}) &= c_{n-2}(V)B_2 \\ &\vdots \\ c_1(B_2) + 3H_V|_{B_2} &= c_1(V)B_2 \end{aligned}$$

Therefore

$$\begin{aligned} c_{n-1}(B_2) &= 3(c_{n-1}(V)H_V + (-3)c_{n-2}(V)H_V^2 \\ &\quad + \cdots + (-3)^{n-2}c_1(V)H^{n-1} + (-3)^{n-1}H^n). \end{aligned} \tag{27}$$

On the other hand, by (22) we have

$$\begin{aligned} c_j(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c_{j-1}(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) + c_j(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) \\ &= 2\binom{n}{j-1}H_V\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{j-1} + \left(\binom{n}{j} - 2\binom{n}{j-1}\right)\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^j. \end{aligned}$$

Hence by using (25) we get

$$\begin{aligned} c_j(V)H^{n-j} &= 2\binom{n}{j-1}H_V^{n-j+1}\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{j-1} + \left(\binom{n}{j} - 2\binom{n}{j-1}\right)H_V^{n-j}\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)^j \\ &= 2^{n-j+1}\binom{n}{j-1} + 2^{n-j-1}\left(\binom{n}{j} - 2\binom{n}{j-1}\right) \\ &= 2^{n-j}\binom{n}{j-1} + 2^{n-j-1}\binom{n}{j}. \end{aligned}$$

Therefore

$$\sum_{j=1}^{n-1}(-3)^{n-j-1}c_j(V)H_V^{n-j} = 2\sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j-1} + \sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j}. \tag{28}$$

On the other hand

$$\begin{aligned} 2\sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j-1} &= \frac{1}{18}\sum_{j=1}^{n-1}(-6)^{n-j+1}\binom{n}{j-1} \\ &= \frac{1}{18}((1+(-6))^n - (-6)n - 1) \\ &= \frac{1}{18}((-5)^n + 6n - 1), \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{n-1}(-6)^{n-j-1}\binom{n}{j} &= -\frac{1}{6}\sum_{j=1}^{n-1}(-6)^{n-j}\binom{n}{j} \\ &= -\frac{1}{6}((1+(-6))^n - (-6)^n - 1) \\ &= \frac{1}{6}((-6)^n - (-5)^n + 1). \end{aligned}$$

Since $H_V^n = 2^{n-1}$, from (27) we get

$$\begin{aligned} & c_{n-1}(B_2) \\ & = 3\left(\frac{1}{18}((-5)^n + 6n - 1) + \frac{1}{6}((-6)^n - (-5)^n + 1) + (-3)^{n-1}2^{n-1}\right) \\ & = -\frac{1}{3}(-5)^n + \frac{3n+1}{3}. \end{aligned} \tag{29}$$

From (18) and (29), we have

$$e(B) = e(B_1) + e(B_2) = 2n + \frac{1-(-5)^n}{3}. \tag{30}$$

By (24) and (30) we get

$$e(X^*) = 2e(V) - e(B) = 2n + \frac{(-5)^n - 1}{3}. \tag{31}$$

Therefore by (17) and (31)

$$e(X) = e(X^*) - e(E) + 1 = n + \frac{(-5)^n + 2}{3}. \tag{32}$$

So we see that

$$e_i(X, L) = i + \frac{(-5)^i + 2}{3} \tag{33}$$

for every integer i with $1 \leq i \leq n$. Here we note that this equality holds for the case where $i = 0$.

Next we calculate $b_i(X,L)$. Since we see from [1, (16.6) 4)] that

$$b_j(X) = \begin{cases} 1, & \text{if } j \text{ is even with } j \leq n-1, \\ 0, & \text{if } j \text{ is odd with } j \leq n-1, \end{cases}$$

we have

$$\begin{aligned} b_i(X,L) &= (-1)^i \left(e_i(X,L) - 2\sum_{j=0}^{i-1} (-1)^j b_j(X) \right) \\ &= (-1)^i \left(i + \frac{(-5)^i + 2}{3} \right) - 2(-1)^i \cdot \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i+1}{2} + 1, & \text{if } i \text{ is odd,} \end{cases} \\ &= \begin{cases} (-1)^i \frac{(-5)^i+2}{3}, & \text{if } i \text{ is even,} \\ (-1)^i \frac{(-5)^i-1}{3}, & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Example 3.8 The case where (X,L) is a hyperquadric fibration over a smooth curve C. Let $f : X \to C$ be its morphism. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projection. Then there exists an embedding $i : X \hookrightarrow \mathbb{P}_C(\mathcal{E})$ such that $f = \pi \circ i$, $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$. Let $e := \deg \mathcal{E}$ and $b := \deg B$. Then by [6, Theorem 3.1], we see that the following holds. Let (X,L) be a hyperquadric fibration over a smooth curve C with $\dim X = n \geq 3$, and let i be an integer with $0 \leq i \leq n$. Then

$$e_i(X,L) = (-1)^i(2e + (i+1)b) + \begin{cases} 2(i+1)(1-g(C)) & \text{if } i \text{ is odd,} \\ 2i(1-g(C)) & \text{if } i \text{ is even.} \end{cases}$$

Example 3.9 The case where (X,L) is a scroll over a smooth curve C.
Then by [4, Corollary 3.1 (3.1.1) and Corollary 3.3 (3.3.1)], we see that the following holds. Let $\mathcal{E}$ be an ample vector bundle of rank n on C such that $X = \mathbb{P}_C(\mathcal{E})$ and $L = H(\mathcal{E})$.

$$e_i(X,L) = \begin{cases} i(2-2g(C)) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

$$b_i(X,L) = \begin{cases} h^i(X,\mathbb{C}) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

Example 3.10 The case where (X,L) is a scroll over a smooth surface S.
Let $\mathcal{E}$ be an ample vector bundle of rank $n-1$ on S such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$. Then by [4, Corollary 3.1 (3.1.2) and Corollary 3.3 (3.3.2)], we see that the following holds.

$$e_i(X,L) = \begin{cases} (i-1)c_2(S) & \text{if } i \geq 3, \\ c_2(S) + c_2(\mathcal{E}) & \text{if } i = 2, \\ -(c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}) & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

$$b_i(X,L) = \begin{cases} h^i(X,\mathbb{C}) & \text{if } m \geq i \geq 3, \\ h^2(X,\mathbb{C}) + c_2(\mathcal{E}) - 1 & \text{if } i = 2, \\ c_1(\mathcal{E})(c_1(\mathcal{E}) + K_S) + 2 & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

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