Correction to "On the sectional geometric genus of quasi-polarized varieties, II", Manuscripta Math. 113, (2004) 211–237.

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In [2, page 44], Harris defined a Castelnuovo variety as follows:

Definition 1 Let X be a projective variety of dimension $n \geq 1$ such that X is a nondegenerate subvariety of $\mathbb{P}^N$. Then X is called a *Castelnuovo variety* if the following conditions are satisfied:

(1) $d \geq n(N-n)+2$.

(2) $h^n(\mathcal{O}_X) = \binom{M}{n+1}(N-n) + \binom{M}{n}\varepsilon$.

Here $d = \deg X$, $M := \left[\frac{d-1}{N-n}\right]$ and $\varepsilon := d-1-M(N-n)$.

Let (X, L) be a polarized manifold such that L is very ample. Then, adopting Definition 1, in the paper [1] we say that (X, L) is a Castelnuovo variety if X is a Castelnuovo variety by the embedding defined by $|L|$.

For a nondegenerate subvariety X of $\mathbb{P}^N$, in [2, Section 5, P.67], in order to prove that the following

($*$) If X is smooth and a hyperplane section X_1 of X is a Castelnuovo variety, then so is X,

Harris proved that X satisfies (2) in Definition 1, but it seems to me that he did not check that X satisfies (1) in Definition 1.

In [1, Theorem 4.7], we use ($*$). From assumptions in [1, Theorem 4.7] we cannot see that X_{n-j} satisfies (1) in Definition 1, that is, $\deg X_{n-j} \geq j(N-(n-j)-j)+2 = j(N-n)+2$ for every j with $i+1 \leq j \leq n$. Hence we don't know (X_{n-j}, L_{n-j}) is a Castelnuovo variety in the above sense, and [1, Theorem 4.7] is incorrect.

So here we would like to modify [1, Theorem 4.7]. Concretely, in [1, Theorem 4.7], we change the statement "Assume that $g_i(X, L) \geq 1$" into "Assume that $d \geq n(N-n)+2$. Namely, Theorem 4.7 is restated as follows:

Theorem 4.7 *Let X be a smooth projective variety of dimension $n \geq 2$ and let L be a very ample line bundle on X. Let $\phi : X \hookrightarrow \mathbb{P}^N$ be the embedding defined by the complete linear system $|L|$. (Here $N = h^0(L) - 1$.) We put $d = L^n$, $M = \left[\frac{d-1}{N-n}\right]$ and $\varepsilon = d - 1 - M(N-n)$. Let i be an integer with $1 \leq i \leq n$. Assume that $d \geq n(N-n) + 2$ and*

$$g_i(X, L) = \binom{M}{i+1}(N-n) + \binom{M}{i}\varepsilon.$$

Then (X, L) is a Castelnuovo variety.

Proof. We use [1, Notation 1.7.1]. First we note that $\phi|_{X_j} : X_j \hookrightarrow \mathbb{P}^{N-j}$ is nondegenerate for every integer j with $0 \leq j \leq n-i$. We prove that X_{n-i-k} is a Castelnuovo variety with respect to $\phi|_{X_{n-i-k}} : X_{n-i-k} \hookrightarrow \mathbb{P}^{N-(n-i-k)}$ for every integer k with $0 \leq k \leq n-i$. From the assumption that $d \geq n(N-n) + 2$, we see that

$$L_{n-i-k}^{i+k} \geq (i+k)(N-n) + 2 = (i+k)(N-(n-i-k)-(i+k)) + 2 \qquad (1)$$

for every integer k with $0 \leq k \leq n-i$. We prove this by induction on k.

Assume that $k = 0$. By assumption

$$\begin{aligned} h^i(\mathcal{O}_{X_{n-i}}) &= g_i(X, L) \\ &= \binom{M}{i+1}(N-n) + \binom{M}{i}\varepsilon \\ &= \binom{M_i}{i+1}\{(N-n+i)-i\} + \binom{M_i}{i}\varepsilon_i, \end{aligned}$$

where $M_i := \left[\frac{d-1}{(N-n+i)-i}\right] = \left[\frac{d-1}{N-n}\right] = M$ and $\varepsilon_i := d - 1 - M_i((N-n+i)-i) = d - 1 - M(N-n) = \varepsilon$. Hence by (1) we see that X_{n-i} is a Castelnuovo variety.

Assume that the assertion holds for $k = t$, where t is an integer with $0 \leq t \leq n-i-1$. Namely we assume that X_{n-i-t} is a Castelnuovo variety with respect to $\phi|_{X_{n-i-t}} : X_{n-i-t} \hookrightarrow \mathbb{P}^{N-(n-i-t)}$. Then by a result of Harris (see [2, Section 5, page 67]), if X_{n-i-t} is a Castelnuovo variety, then so is $X_{n-i-t-1}$ by (1). Hence by induction we get the assertion. □

References

[1] Y. Fukuma, *On the sectional geometric genus of quasi-polarized varieties, II,* Manuscripta Math. 113 (2004), 211–237.

[2] J. Harris, *A bound on the geometric genus of projective varieties,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser. 8 (1981), 35–68.

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