Invariants of ample line bundles on projective varieties and their applications, III *†‡

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Abstract

Let X be a smooth complex projective variety of dimension n and let $L_1, \dots, L_{n-i}$ be ample line bundles on X, where i is an integer with $0 \leq i \leq n-1$. In the first part, we defined the ith sectional geometric genus $g_i(X, L_1, \dots, L_{n-i})$ and the ith sectional H-arithmetic genus $\chi_i^H(X, L_1, \dots, L_{n-i})$ of $(X, L_1, \dots, L_{n-i})$. In this third part, we will investigate $g_2(X, L_1, \dots, L_{n-2})$ and $\chi_2^H(X, L_1, \dots, L_{n-2})$. Moreover we will give some applications of the sectional invariants of multi-polarized manifolds.

Introduction.

Let X be a projective variety of dimension n which is defined over the field of complex numbers and let L be an ample (resp. nef and big) line bundle on X. Then the pair (X, L) is called a *polarized* (*resp. quasi-polarized*) *variety*. Moreover if X is smooth, then (X, L) is called a polarized (resp. quasi-polarized) *manifold*.

This is the continuation of [11] and [12]. The third part consists of Sections 7, 8 and 9. Let X be a smooth complex projective variety of dimension n and let $L_1, \dots, L_{n-i}$ be ample line bundles on X, where i is an integer with $0 \leq i \leq n-1$. In Section 7 we will give some results and definitions which will be used in this paper. In Section 8 we will deal with the second sectional invariants of multi-polarized manifolds $(X, L_1, \dots, L_{n-2})$. By using the sectional invariants of $(X, L_1, \dots, L_{n-2})$ we can get some statements for multi-polarized manifolds which are considered to be a kind of generalization of well-known results in the theory of projective surfaces. In particular, we will give two problems which are multi-polarized manifolds' version of Castelnuovo's theorem and Bogomolov-Miyaoka-Yau's theorem, and we will investigate these. In Section 9, we will give two applications in this paper. In [10] and [13], we gave an application of sectional geometric genus of multi-polarized manifolds to calculation of the dimension of the global sections of adjoint bundles. As another application, first, we will calculate the sectional geometric genus of complete intersections of hypersurfaces in the projective space by using the sectional geometric genus of multi-polarized manifolds. Next we will give the definition of the *ith sectional m-genus* of multi-quasi-polarized manifolds, which is thought to be a generalization of the m-genus of minimal projective variety of general type. Also we will investigate this invariant, and we can get some results for $i = 1$ and 2 which are considered to be a generalization of results in the theory of curves and surfaces. Here we note that we cannot define the ith sectional m-genus of quasi-polarized manifold easily without the notion of the ith sectional geometric genus of multi-quasi-polarized manifolds.

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* *Key words and phrases.* Polarized varieties, quasi-polarized varieties, ample line bundles, nef and big line bundles, sectional genus, ith sectional geometric genus, ith sectional H-arithmetic genus, adjoint bundles.

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7 Preliminaries for the third part

Definition 7.1 Let X be a projective variety and let L be a line bundle on X. Then L is said to be *k-big* if $\kappa(L) \geq \dim X - k$, where k is an integer with $0 \leq k \leq \dim X$.

Proposition 7.1 (Generalized Hodge Index Theorem) *Let X be a projective variety of dimension n, let k be a natural number and let L_i be a line bundle on X for $0 \leq i \leq k$. Assume that $n \geq 2$ and L_i is nef for $i \geq 1$. If $n_1 + \cdots + n_k = n-1$ and $n_1 \geq 1$, then we have*

$$(L_0 L_1^{n_1} L_2^{n_2} \cdots L_k^{n_k})^2 \geq (L_0^2 L_1^{n_1-1} L_2^{n_2} \cdots L_k^{n_k})(L_1^{n_1+1} L_2^{n_2} \cdots L_k^{n_k}).$$

Proof. See [1, Proposition 2.5.1]. □

Proposition 7.2 *Let a polarized manifold (X, L) be a quadric fibration over a normal variety Y. Set $n := \dim X$ and $m := \dim Y$. Then the intersection number $f^*(N_1) \cdots f^*(N_m) L^{n-m}$ is even for every line bundles $N_1, \ldots, N_m$ on Y.*

Proof. Let p be a positive integer such that pL is very ample. By Bertini's theorem, there exists an m-dimensional projective variety T such that T is an intersection of $(n-m)$ general members of $|pL|$. Then $f|_T : T \to Y$ is a surjective morphism with $\deg f_T = 2p^{n-m}$ and

$$\begin{aligned} f^*(N_1) \cdots f^*(N_m)(pL)^{n-m} &= f^*(N_1) \cdots f^*(N_m) T \\ &= (f^*(N_1))|_T \cdots (f^*(N_m))|_T \\ &= (f|_T)^*(N_1)) \cdots (f|_T)^*(N_m) \\ &= (\deg f|_T) N_1 \cdots N_m \\ &= 2p^{n-m} N_1 \cdots N_m. \end{aligned}$$

Therefore we get the assertion. □

Lemma 7.1 *Let X be a complete normal variety, and let D_1 and D_2 be effective Cartier divisors on X. Then $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$.*

Proof. See [5, Lemma 1.12] or [16, 15.6.2 Lemma]. □

Theorem 7.1 *Let X be a smooth projective variety of dimension $n \geq 3$, and let $H_1, \ldots, H_{n-2}$ be ample Cartier divisors on X. Let B be an ample $\mathbb{Q}$-Cartier divisor on X such that $K_X + nB$ is nef and $(n-2)$-big. Assume that $\kappa(X) \geq 0$. Then*

$$c_2(X) H_1 \cdots H_{n-2} \geq -(n-1) K_X B H_1 \cdots H_{n-2} - \binom{n}{2} B^2 H_1 \cdots H_{n-2}.$$

Proof. See [8, Theorem 2.1] and [18, Corollary 6.4]. □

Notation 7.1 Let X be a smooth projective variety of dimension n and let i be an integer with $1 \leq i \leq n-1$. Let $L_1, \ldots, L_{n-i}$ be nef and big line bundles on X. Assume that $\mathrm{Bs}|L_j| = \emptyset$ for every integer j with $1 \leq j \leq n-i$. Then by Bertini's theorem, for every integer j with $1 \leq j \leq n-i$, there exists a general member $X_j \in |L_j|_{X_{j-1}}|$ such that X_j is a smooth projective variety of dimension $n-j$. (Here we set $X_0 := X$.)

8 The second sectional invariants

8.1 The second sectional geometric genus

Proposition 8.1.1 *Let X be a smooth projective variety of dimension $n \geq 3$. Let $L_1, \dots, L_{n-2}$ be line bundles on X. Then*

$$\begin{aligned} & g_2(X, L_1, \dots, L_{n-2}) \\ & = -1 + h^1(\mathcal{O}_X) + \frac{1}{6}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} + \frac{1}{4}\left(\sum_{1 \leq j < k \leq n-2} L_j L_k\right) L_1 \cdots L_{n-2} \\ & \quad + \frac{1}{4} K_X \left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} + \frac{1}{12}(c_2(X) + K_X^2) L_1 \cdots L_{n-2}. \end{aligned}$$

Proof. We use [11, Corollary 2.7] for $i = 2$. Here we see from the proof of [11, Theorem 2.4] that the equality in [11, Corollary 2.7] is true for any line bundles $L_1, \dots, L_{n-i}$.
By [11, Corollary 2.7], we can describe $g_2(X, L_1, \dots, L_{n-2})$ by using the following four terms[1]

$$\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2}, \quad \left(\sum_{1 \leq j < k \leq n-2} L_j L_k\right) L_1 \cdots L_{n-2}$$

$$\left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} T_1(X) \quad \text{and} \quad L_1 \cdots L_{n-2} T_2(X).$$

The coefficient of the above four terms are the following.

$\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2}$	$\left(\sum_{1 \leq j < k \leq n-2} L_j L_k\right) L_1 \cdots L_{n-2}$
$(-1)^0/(3!1! \cdots 1!) = 1/6$	$(-1)^0/(2!2!1! \cdots 1!) = 1/4$

$\left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} T_1(X)$	$L_1 \cdots L_{n-2} T_2(X)$
$(-1)^1/(2!1! \cdots 1!) = -1/2$	$(-1)^2/(1! \cdots 1!) = 1$

Since $T_1(X) = (1/2)c_1(X) = -(1/2)K_X$ and $T_2(X) = (1/12)(c_2(X) + c_1(X)^2)$, we obtain the assertion. □

Theorem 8.1.1 *Let X be a smooth projective variety of dimension $n \geq 4$. Let $L_1, \dots, L_{n-2}$ be ample and spanned line bundles on X. If $g_2(X, L_1, \dots, L_{n-2}) = h^2(\mathcal{O}_X)$, then $(X, L_{\sigma(1)}, \dots, L_{\sigma(n-2)})$ is one of the following: (Here σ is an element of the symmetric group $\mathfrak{S}_{n-2}$ of $\{1, \dots, n-2\}$.)*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) *$n \geq 5$ and* $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(2), \mathcal{O}_{\mathbb{P}^n}(2))$.

(3) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(2))$.

(4) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(3))$.

(5) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1), \dots, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(6) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1), \dots, \mathcal{O}_{\mathbb{Q}^n}(1), \mathcal{O}_{\mathbb{Q}^n}(2))$.

[1] For the definition of $T_k(X)$, see [11, Definition 1.7].

(7) *X is a $\mathbb{P}^{n-1}$-bundle over a smooth curve C and one of the following holds.* (*Here F denotes its fiber*).

(7.1) $L_{\sigma(j)}|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ *for every integer j with $1 \leq j \leq n-2$.*

(7.2) $L_{\sigma(j)}|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ *for every integer j with $1 \leq j \leq n-3$ and $L_{\sigma(n-2)}|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(2)$.*

(8) *$K_X + (n-1)L_j = \mathcal{O}_X$ for any j. In particular $L_j = L_k$ for any (j,k) with $j \neq k$.*

(9) *There exist a smooth projective curve W and a surjective morphism $f : X \to W$ with connected fibers such that (X, L_i) is a quadric fibration over W with respect to f for every integer i with $1 \leq i \leq n-2$.*

(10) *There exist a smooth projective surface S and a surjective morphism $f : X \to S$ with connected fibers such that f is a $\mathbb{P}^{n-2}$-bundle over S and (X, L_j) is a scroll over S with respect to f for every integer j with $1 \leq j \leq n-2$.*

(11) *Let $(Y, A_1, \ldots, A_{n-2})$ be a reduction of $(X, L_1, \ldots, L_{n-2})$. Then $n = 4$ and $(Y, A_1, A_2) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2), \mathcal{O}_{\mathbb{P}^4}(2))$.*

Proof. Here we use notation in Notation 7.1. Let $((X_{n-3})', (L_{n-2}|_{X_{n-3}})')$ be a reduction of $(X_{n-3}, L_{n-2}|_{X_{n-3}})$. Since

$$g_2(X_{n-3}, L_{n-2}|_{X_{n-3}}) = g_2(X, L_1, \ldots, L_{n-2}) = h^2(\mathcal{O}_X) = h^2(\mathcal{O}_{X_{n-3}}),$$

$K_{(X_{n-3})'} + (L_{n-2}|_{X_{n-3}})'$ is not nef by [5, Corollary 3.5].

Here we note that $K_{X_{n-3}} + L_{n-2}|_{X_{n-3}}$ is not nef. (If $K_{X_{n-3}} + L_{n-2}|_{X_{n-3}}$ is nef, then $K_{X_{n-3}} + 2L_{n-2}|_{X_{n-3}}$ is ample. Hence $(X_{n-3}, L_{n-2}|_{X_{n-3}}) \cong ((X_{n-3})', (L_{n-2}|_{X_{n-3}})')$. But this is impossible because $K_{(X_{n-3})'} + (L_{n-2}|_{X_{n-3}})'$ is not nef.)

Hence $K_X + L_1 + \cdots + L_{n-2}$ is not nef. By [12, Remark 5.2.4], $(X, L_{\sigma(1)}, \ldots, L_{\sigma(n-2)})$ is one of the types above.

If $(X, L_{\sigma(1)}, \ldots, L_{\sigma(n-2)})$ is one of the above, then we can easily see that

$$g_2(X, L_1, \ldots, L_{n-2}) = h^2(\mathcal{O}_X).$$

Hence we get the assertion. □

Remark 8.1.1 (1) Theorem 8.1.1 has been also obtained by Lanteri [17, Theorems (3.2) and (3.3)] (see also [10, Remark 3.1 (3)]).
(2) We can easily check that (X, L_1, L_2) in (11) of Theorem 8.1.1 is a simple blowing up of (Y, A_1, A_2).
(3) In [17] Lanteri showed that if $(X, L_1, \ldots, L_{n-2})$ is the type of (8) in Theorem 8.1.1, then for any i with $1 \leq i \leq n-2$ we see that $2 \leq L_i^4 \leq 6$ (resp. $2 \leq L_i^n \leq 5$) if $n = 4$ (resp. $n \geq 5$).

Theorem 8.1.2 *Let X be a smooth projective variety of dimension $n \geq 3$. Let $L_1, \ldots, L_{n-2}$ be ample line bundles on X. Assume that $\kappa(X) \geq 0$. Then*

$$\begin{aligned} g_2(X, L_1, \ldots, L_{n-2}) \quad &\geq \quad -1 + h^1(\mathcal{O}_X) + \frac{1}{24}(L_1^2 + \cdots + L_{n-2}^2)L_1 \cdots L_{n-2} \\ &\quad + \frac{1}{24}\left(2 + \frac{1}{n}\right)(L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2}. \end{aligned}$$

Proof. By taking a reduction, if necessary, we may assume that $K_X + L_1 + \cdots + L_{n-2}$ is nef and $(n-2)$-big by [12, Remark 5.2.4] and [12, Theorem 5.2.3] because $\kappa(X) \geq 0$. By Theorem 7.1 we get the following lower bound

$$c_2(X)L_1 \cdots L_{n-2} \geq -(n-1)K_X B L_1 \cdots L_{n-2} - \binom{n}{2} B^2 L_1 \cdots L_{n-2},$$

where

$$B = \frac{1}{n}(L_1 + \cdots + L_{n-2}).$$

So by Proposition 8.1.1, we obtain

$$\begin{aligned}
& g_2(X, L_1, \ldots, L_{n-2}) \\
& \geq -1 + h^1(\mathcal{O}_X) + \frac{1}{6}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} + \frac{1}{4}\left(\sum_{1 \leq j < k \leq n-2} L_j L_k\right) L_1 \cdots L_{n-2} \\
& \quad + \frac{1}{4} K_X \left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} + \frac{1}{12} K_X^2 L_1 \cdots L_{n-2} \\
& \quad - \frac{n-1}{12n} K_X L_1 \cdots L_{n-2} \left(\sum_{j=1}^{n-2} L_j\right) - \frac{n-1}{24n} L_1 \cdots L_{n-2} \left(\sum_{j=1}^{n-2} L_j\right)^2 \\
& = -1 + h^1(\mathcal{O}_X) + \frac{1}{12} K_X (K_X + L_1 + \cdots + L_{n-2}) L_1 \cdots L_{n-2} \\
& \quad + \frac{1}{12}\left(1 + \frac{1}{n}\right) K_X (L_1 + \cdots + L_{n-2}) L_1 \cdots L_{n-2} \\
& \quad + \frac{1}{24}\left(2 + \frac{1}{n}\right) (L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\
& \quad + \frac{1}{24}(L_1^2 + \cdots + L_{n-2}^2) L_1 \cdots L_{n-2} \\
& \geq -1 + h^1(\mathcal{O}_X) + \frac{1}{24}\left(2 + \frac{1}{n}\right) (L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\
& \quad + \frac{1}{24}(L_1^2 + \cdots + L_{n-2}^2) L_1 \cdots L_{n-2}.
\end{aligned}$$

So we get the assertion. □

8.2 The second sectional *H*-arithmetic genus

Let n be an integer with $n \geq 3$. Let $(X, L_1, \ldots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $(n-2)$. Here we are going to propose some conjectures which are induced by some results in the surface theory. Here we note the following: let $(X, L_1, \ldots, L_{n-2})$ be a multi-polarized manifold of type $(n-2)$ with $\dim X = n$. Assume that $\mathrm{Bs}|L_j| = \emptyset$ for every integer j with $1 \leq j \leq n-2$, and $(X, L_1, \ldots, L_{n-2})$ is not the type (10) in [12, Remark 5.2.4]. Then by the same argument as in the proof of [9, Proposition 2.1], we can prove the following: (Here we use Notation 7.1.)

(a) $\kappa(K_X + L_1 + \cdots + L_{n-2}) \geq 2$ if and only if $\kappa(X_{n-2}) = 2$.

(b) $\kappa(K_X + L_1 + \cdots + L_{n-2}) = 1$ if and only if $\kappa(X_{n-2}) = 1$.

(c) $\kappa(K_X + L_1 + \cdots + L_{n-2}) = 0$ if and only if $\kappa(X_{n-2}) = 0$.

(d) $\kappa(K_X + L_1 + \cdots + L_{n-2}) = -\infty$ if and only if $\kappa(X_{n-2}) = -\infty$.

By the same consideration as in [6] and [9], we can give the following correspondences: let S be a smooth projective surface. Then the following correspondences are considered.

Invariants of S.	$\Leftrightarrow$	Invariants of $(X, L_1, \dots, L_{n-2})$.
$h^2(\mathcal{O}_S)$	$\Leftrightarrow$	$g_2(X, L_1, \dots, L_{n-2})$
$h^1(\mathcal{O}_S)$	$\Leftrightarrow$	$h^1(\mathcal{O}_X)$
$\chi(\mathcal{O}_S)$	$\Leftrightarrow$	$\chi_2^H(X, L_1, \dots, L_{n-2})$
K_S^2	$\Leftrightarrow$	$(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2}$
$\kappa(S) = k$	$\Leftrightarrow^*$	$\kappa(K_X + L_1 + \cdots + L_{n-2}) = k$
$\kappa(S) = 2$	$\Leftrightarrow^{**}$	$\kappa(K_X + L_1 + \cdots + L_{n-2}) \geq 2$

(In $(*)$, $k = -\infty, 0$, or 1. In $(*)$ and $(**)$, the direction $\Rightarrow$ needs the assumption that $(X, L_1, \dots, L_{n-2})$ is not the type (10) in [12, Remark 5.2.4].)

By using these correspondences, we can propose many problems. Before we propose conjectures, we state the following some fundamental results in the surface theory.

(A) $\chi(\mathcal{O}_S) > 0$ if $\kappa(S) = 2$.

(B) $\chi(\mathcal{O}_S) \geq 0$ if $0 \leq \kappa(S) \leq 1$.

(C) $\chi(\mathcal{O}_S) = 1 - q(S)$ if $\kappa(S) = -\infty$.

(D) $8\chi(\mathcal{O}_S) \geq K_S^2$ if $\kappa(S) = -\infty$ and S is not isomorphic to $\mathbb{P}^2$.

(E) $9\chi(\mathcal{O}_S) \geq K_S^2$ if $\kappa(S) \geq 0$.

By using the above correspondences, we can propose the following conjecture.

Conjecture 8.2.1 *Let $(X, L_1, \dots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $(n-2)$. Then*

(1) $\chi_2^H(X, L_1, \dots, L_{n-2}) > 0$ *if* $\kappa(K_X + L_1 + \cdots + L_{n-2}) \geq 2$.

(2) $\chi_2^H(X, L_1, \dots, L_{n-2}) \geq 0$ *if* $0 \leq \kappa(K_X + L_1 + \cdots + L_{n-2}) \leq 1$.

(3) $\chi_2^H(X, L_1, \dots, L_{n-2}) = 1 - q(X)$ *if* $\kappa(K_X + L_1 + \cdots + L_{n-2}) = -\infty$ *and* $(X, L_1, \dots, L_{n-2})$ *is not the type* (10) *in* [12, Remark 5.2.4].

(4) $8\chi_2^H(X, L_1, \dots, L_{n-2}) \geq (K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2}$ *if* $\kappa(K_X + L_1 + \cdots + L_{n-2}) = -\infty$ *and* $(X, L_1, \dots, L_{n-2})$ *is neither* $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1))$ *nor the type* (10) *in* [12, Remark 5.2.4].

(5) $9\chi_2^H(X, L_1, \dots, L_{n-2}) \geq (K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2}$ *if* $\kappa(K_X + L_1 + \cdots + L_{n-2}) \geq 0$.

If $n = 3$, then this conjecture is equivalent to [9, Conjecture 2.1]. So in this paper we consider the case where $n \geq 4$. First we will study (2) in Conjecture 8.2.1.

Theorem 8.2.1 *Let n be an integer with $n \geq 4$. Let $(X, L_1, \dots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $n-2$. Assume that $\kappa(K_X + L_1 + \cdots + L_{n-2}) = 0$ or 1. Then $\chi_2^H(X, L_1, \dots, L_{n-2}) > 0$.*

Proof. By taking a reduction of $(X, L_1, \cdots, L_{n-2})$ if necessary, we may assume that the multi-polarized manifold $(X, L_1, \cdots, L_{n-2})$ is a reduction of itself (see [11, Proposition 2.3]). Then $K_X + L_1 + \cdots + L_{n-2}$ is nef but not ample. Let $\Phi : X \to W$ be the nef value morphism of $(X, L_1 + \cdots + L_{n-2})$. Then $\dim W = 0$ (resp. $\dim W = 1$) if $\kappa(K_X + L_1 + \cdots + L_{n-2}) = 0$ (resp. 1).

If $\dim W = 0$, then $K_X + L_1 + \cdots + L_{n-2} = \mathcal{O}_X$. Hence $h^0(K_X + L_1 + \cdots + L_{n-2}) = 1$ and $h^1(\mathcal{O}_X) = 0$. Therefore by [11, Example 2.1 (G)] we have $g_2(X, L_1, \dots, L_{n-2}) = 1$ and $\chi_2^H(X, L_1, \dots, L_{n-2}) = 2$.

If $\dim W = 1$, then $(X, L_1, \cdots, L_{n-2})$ is a Del Pezzo fibration over a smooth curve W. In this case $K_X + L_1 + \cdots + L_{n-2} = \Phi^*(H)$ for an ample line bundle H on W. Then

$$\begin{aligned} g_2(X, L_1, \cdots, L_{n-2}) &= h^0(K_X + L_1 + \cdots + L_{n-2}) \\ &= h^0(H) \\ &= h^1(H) + 1 - g(W) + \deg(H). \end{aligned}$$

Since $\deg(H) > 2g(W) - 2$ by [5, Lemma 1.13 (2)], we have

$$\begin{aligned} g_2(X, L_1, \cdots, L_{n-2}) &= h^1(H) + 1 - g(W) + \deg(H) \\ &> h^1(H) + 1 - g(W) + 2g(W) - 2 \\ &= h^1(H) + g(W) - 1. \end{aligned}$$

Therefore, since $g(W) = h^1(\mathcal{O}_X)$,

$$\begin{aligned} &\chi_2^H(X, L_1, \cdots, L_{n-2}) \\ &= 1 - h^1(\mathcal{O}_X) + g_2(X, L_1, \cdots, L_{n-2}) \\ &> h^1(H) \geq 0. \end{aligned}$$

This completes the proof. □

Next we consider (1) in Conjecture 8.2.1 for $\kappa(X) \geq 0$.

Theorem 8.2.2 *Let $(X, L_1, \ldots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $n-2$. Assume that $n \geq 4$ and $\kappa(X) \geq 0$. Then*

$$\begin{aligned} \chi_2^H(X, L_1, \ldots, L_{n-2}) &\geq \frac{1}{24}(L_1^2 + \cdots + L_{n-2}^2)L_1 \cdots L_{n-2} \\ &\quad + \frac{1}{24}\left(2 + \frac{1}{n}\right)(L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\ &> 0. \end{aligned}$$

Proof. Since

$$\chi_2^H(X, L_1, \ldots, L_{n-2}) = 1 - h^1(\mathcal{O}_X) + g_2(X, L_1, \ldots, L_{n-2}),$$

we get the assertion by Theorem 8.1.2. □

Next we consider (3) and (4) in Conjecture 8.2.1.

Theorem 8.2.3 *Let n be an integer with $n \geq 4$. Let $(X, L_1, \ldots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $n-2$. Assume that $\kappa(K_X + L_1 + \cdots + L_{n-2}) = -\infty$. Then the following hold.*

(1) $\chi_2^H(X, L_1, \ldots, L_{n-2}) = 1 - q(X)$ *if $(X, L_1, \ldots, L_{n-2})$ is not the type* (10) *in* [12, Remark 5.2.4].

(2) $8\chi_2^H(X, L_1, \ldots, L_{n-2}) \geq (K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2}$ *if $(X, L_1, \ldots, L_{n-2})$ is neither $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(1))$ nor the type* (10) *in* [12, Remark 5.2.4].

(3) *If $(X, L_1, \ldots, L_{n-2}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(1))$, then*

$$9\chi_2^H(X, L_1, \ldots, L_{n-2}) = (K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} = 9.$$

Proof. Assume that $\kappa(K_X+L_1+\cdots+L_{n-2})=-\infty$ and $(X,L_1,\ldots,L_{n-2})$ is not the type (10) in [12, Remark 5.2.4]. Then $K_X+L_1+\cdots+L_{n-2}$ is not nef, and $(X,L_1,\ldots,L_{n-2})$ is one of the types in [12, Remark 5.2.4] other than the type (10) in [12, Remark 5.2.4]. Here, by using [11, Corollary 2.3], we calculate $g_2(X,L_1,\cdots,L_{n-2})$, $\chi_2^H(X,L_1,\cdots,L_{n-2})$ and $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}$.

(a) If $(X,L_1,\cdots,L_{n-3},L_{n-2})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1),\cdots,\mathcal{O}_{\mathbb{P}^n}(1),\mathcal{O}_{\mathbb{P}^n}(3))$, then $g_2(X,L_1,\cdots,L_{n-2})=0$, $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=3$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1=1-q(X)$.

(b) If $(X,L_1,\cdots,L_{n-4},L_{n-3},L_{n-2})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1),\cdots,\mathcal{O}_{\mathbb{P}^n}(1),\mathcal{O}_{\mathbb{P}^n}(2),\mathcal{O}_{\mathbb{P}^n}(2))$, then we see that $g_2(X,L_1,\cdots,L_{n-2})=0$, $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=4$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1=1-q(X)$.

(c) If $(X,L_1,\cdots,L_{n-3},L_{n-2})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1),\cdots,\mathcal{O}_{\mathbb{P}^n}(1),\mathcal{O}_{\mathbb{P}^n}(2))$, then $g_2(X,L_1,\cdots,L_{n-2})=0$, $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=8$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1=1-q(X)$.

(d) If $(X,L_1,\cdots,L_{n-2})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1),\cdots,\mathcal{O}_{\mathbb{P}^n}(1))$, then $g_2(X,L_1,\cdots,L_{n-2})=0$, $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=9$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1=1-q(X)$. Hence $9\chi_2^H(X,L_1,\ldots,L_{n-2})=(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}$.

(e) If $(X,L_1,\cdots,L_{n-3},L_{n-2})\cong(\mathbb{Q}^n,\mathcal{O}_{\mathbb{Q}^n}(1),\cdots,\mathcal{O}_{\mathbb{Q}^n}(1),\mathcal{O}_{\mathbb{Q}^n}(2))$, then $g_2(X,L_1,\cdots,L_{n-2})=0$, $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=4$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1=1-q(X)$.

(f) If $(X,L_1,\cdots,L_{n-2})\cong(\mathbb{Q}^n,\mathcal{O}_{\mathbb{Q}^n}(1),\cdots,\mathcal{O}_{\mathbb{Q}^n}(1))$, then $g_2(X,L_1,\cdots,L_{n-2})=0$, $(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=8$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1=1-q(X)$.

(g) Let $(X,L_1,\ldots,L_{n-2})$ be the type (9) in [12, Remark 5.2.4]. Let $f:X\to W$ be its morphism. Then $K_X+L_1+\cdots+L_{n-2}+L_i=f^*(A_i)$ and $K_X+L_1+\cdots+L_{n-2}+L_j=f^*(A_j)$ for any i and j with $i\neq j$, where $A_k\in\operatorname{Pic}(W)$ for any k. Therefore $L_i-L_j=f^*(A_i-A_j)$, that is, $L_i=L_j+f^*(A_i-A_j)$. Then $K_X+L_1+\cdots+L_{n-2}=K_X+(n-2)L_1+f^*(A_2+\cdots+A_{n-2}-(n-3)A_1)$. We may assume that $\deg(A_i)\geq\deg A_1$ for any i. Here we set $A:=A_2+\cdots+A_{n-2}-(n-3)A_1$. Then $\deg A\geq 0$. Here we note that (X,L_1) is a quadric fibration over W. Let $\mathcal{E}_1:=f_*(L_1)$. Then $\mathcal{E}_1$ is a locally free sheaf of rank $n+1$ and $X\in|2H(\mathcal{E}_1)+p^*(B)|$ for some $B\in\operatorname{Pic}(W)$, where $p:\mathbb{P}_W(\mathcal{E}_1)\to W$, and $K_X=-(n-1)L_1+f^*(K_W+c_1(\mathcal{E}_1)+B)$. We set $a:=\deg A$, $a_i:=\deg A_i$, $b:=\deg B$ and $e:=\deg c_1(\mathcal{E}_1)$. Here we note that $a=\sum_{i=2}^{n-2}a_i-(n-3)a_1$ and $L_1^n=2e+b>0$. Then

$$\begin{aligned}
&(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}\\
&=(-L_1+(2g(W)-2+e+a+b)F)^2L_1(L_1+(a_2-a_1)F)\cdots(L_1+(a_{n-2}-a_1)F)\\
&=L_1^n-2(2g(W)-2+e+a+b)L_1^{n-1}F+(a_2+\cdots+a_{n-2}-(n-3)a_1)L_1^{n-1}F\\
&=8-8g(W)-2e-2a-3b.
\end{aligned}$$

Claim 8.2.1 $2e+3b>0$.

Proof. If $b\geq 0$, then $2e+3b=(2e+b)+2b>0$. So we assume that $b<0$. Since $2e+(n+1)b\geq 0$ by [2, (3.3)] and $n\geq 4$, we see that $2e+3b\geq-(n-2)b>0$. Therefore we get the assertion. □

Since $g_2(X,L_1,\cdots,L_{n-2})=0$ and $\chi_2^H(X,L_1,\cdots,L_{n-2})=1-g(W)$ by [11, Example 2.1 (I)], we get

$$(K_X+L_1+\cdots+L_{n-2})^2L_1\cdots L_{n-2}=8-8g(W)-2e-2a-3b<8(1-g(W))=8\chi_2^H(X,L_1,\ldots,L_{n-2})$$

because $a\geq 0$ and $2e+3b>0$.

(h) Assume that $K_X+(n-1)L_i=\mathcal{O}_X$ for every i with $1\leq i\leq n-2$ and $L_i=L_j$ for $i\neq j$. Then by [11, Example 2.1 (G)], we have $g_2(X, L_1, \cdots, L_{n-2}) = 0$ and $\chi_2^H(X, L_1, \cdots, L_{n-2}) = 1$. Here we note that by [4, (8.11) Theorem] $L_i^n \leq 8$ for every i because (X, L_i) is a Del Pezzo manifold for every i. Then

$$(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} = (K_X + (n-2)L_i)^2 L_i^{n-2} = L_i^n \leq 8 = 8\chi_2^H(X, L_1, \dots, L_{n-2}).$$

(i) Let $(X, L_1, \dots, L_{n-2})$ be the type (7.1) in [12, Remark 5.2.4]. Then by [11, Example 2.1 (H)], we have $g_2(X, L_1, \cdots, L_{n-2}) = 0$ and $\chi_2^H(X, L_1, \cdots, L_{n-2}) = 1 - g(W)$. Let $\Phi : X \to W$ be a $\mathbb{P}^{n-1}$-bundle over W such that $L_i|_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for every i. Then we may assume that there exists an ample vector bundle $\mathcal{E}$ on W such that $L_1 = H(\mathcal{E})$ and there exists $B_j \in \mathrm{Pic}(W)$ such that $L_j = L_1 + \Phi^*(B_j)$ for every j with $2 \leq j \leq n-2$. We set $b_j := \deg B_j$ and $e := c_1(\mathcal{E})$. Then

$$\begin{aligned}
&(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\
&= \left(-2H(\mathcal{E}) + \Phi^*\left(K_W + c_1(\mathcal{E}) + \sum_{j=2}^{n-2} B_j\right)\right)^2 H(\mathcal{E}) \prod_{j=2}^{n-2} (H(\mathcal{E}) + \Phi^*(B_j)) \\
&= 4H(\mathcal{E})^n + 4\left(\sum_{j=2}^{n-2} b_j\right) - 4\left(2g(W) - 2 + e + \sum_{j=2}^{n-2} b_j\right) \\
&= 8(1 - g(W)) \\
&= 8\chi_2^H(X, L_1, \dots, L_{n-2}).
\end{aligned}$$

(j) Let $\Phi : X \to W$ be a $\mathbb{P}^{n-1}$-bundle over a smooth curve W such that $L_i|_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for every i with $1 \leq i \leq n-3$ and $L_{n-2}|_F \cong \mathcal{O}_{\mathbb{P}^{n-1}}(2)$. Then $(X, K_X + L_1 + \cdots + L_{n-3} + 2L_{n-2})$ is a scroll over W. We set $\mathcal{E} := \Phi_*(K_X + L_1 + \cdots + L_{n-3} + 2L_{n-2})$. Then $\mathcal{E}$ is a locally free sheaf of rank n on W such that $X = \mathbb{P}_W(\mathcal{E})$ and $K_X + L_1 + \cdots + L_{n-3} + 2L_{n-2} = H(\mathcal{E})$. On the other hand, we can express L_i as $L_i = H(\mathcal{E}) + \Phi^*(A_i)$ and $L_{n-2} = 2H(\mathcal{E}) + \Phi^*(B)$, where $A_i, B \in \mathrm{Pic}(W)$. We set $a_i := \deg A_i$ for every integer i with $1 \leq i \leq n-3$ and $b := \deg B$. We calculate $(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2}$.

$$\begin{aligned}
&(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\
&= (-H(\mathcal{E}) + \Phi^*(K_W + c_1(\mathcal{E}) + A_1 + \cdots + A_{n-3} + B))^2 \\
&\quad \times (H(\mathcal{E}) + \Phi^*(A_1)) \cdots (H(\mathcal{E}) + \Phi^*(A_{n-3}))(2H(\mathcal{E}) + \Phi^*(B)) \\
&= 2e - 4(2g(W) - 2) - 4e - 4\left(\sum_{i=1}^{n-3} a_i\right) - 4b + 2\left(\sum_{i=1}^{n-3} a_i\right) + b \\
&= -2e + 8(1 - g(W)) - 2\left(\sum_{i=1}^{n-3} a_i\right) - 3b.
\end{aligned}$$

On the other hand

$$\begin{aligned}
0 &< (L_1 \cdots L_{n-3}) L_{n-2}^3 \\
&= (H(\mathcal{E}) + \Phi^*(A_1)) \cdots (H(\mathcal{E}) + \Phi^*(A_{n-3}))(2H(\mathcal{E}) + \Phi^*(B))^3 \\
&= 8\left(\sum_{i=1}^{n-3} a_i\right) + 8e + 12b.
\end{aligned}$$

Hence $2(\sum_{i=1}^{n-3} a_i) + 2e + 3b > 0$. Since $(X, L_1, \dots, L_{n-3}, L_{n-3}, L_{n-2})$ is a quadric fibration over W, by [11, Example 2.1 (I)], we have

$$g_2(X, L_1, \cdots, L_{n-2}) = 0$$

and

$$\chi_2^H(X, L_1, \cdots, L_{n-2}) = 1 - g(W).$$

Therefore we obtain

$$\begin{aligned}
&8\chi_2^H(X, L_1, \dots, L_{n-2}) - (K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\
&= 2\left(\sum_{i=1}^{n-3} a_i\right) + 2e + 3b \\
&> 0.
\end{aligned}$$

By above we get the assertion of Theorem 8.2.3. □

Next we consider the case where $(X, L_1, \dots, L_{n-2})$ is the type (10) in [12, Remark 5.2.4]. Namely, assume that there exist a smooth projective surface S and a surjective morphism $\pi : X \to S$ with connected fibers such that π is a $\mathbb{P}^{n-2}$-bundle over S and (X, L_j) is a scroll over S with respect to f for every integer j with $1 \leq j \leq n-2$. For every integer i with $0 \leq i \leq n-2$, let t_i be a non-negative integer with $t_1 + \cdots + t_{n-2} = n-2$. Then we set

$$f(t_1, \dots, t_{n-2}) := (K_X + t_1 L_1 + \cdots + t_{n-2} L_{n-2})^2 L_1^{t_1} \cdots L_{n-2}^{t_{n-2}}$$

and

$$A_{n-2} := \left\{ (a_1, \dots, a_{n-2}) \;\middle|\; 0 \leq a_i \in \mathbb{Z},\ \sum_{j=1}^{n-2} a_j = n-2 \right\}.$$

Lemma 8.2.1 *Let i and j be two distinct natural numbers such that $1 \leq i, j \leq n-2$ and $L_i \not\equiv L_j$. Then for every $(a_1, \dots, a_{n-2})$, $(b_1, \dots, b_{n-2})$, $(c_1, \dots, c_{n-2}) \in A_{n-2}$ with*

$$\begin{cases} b_i = a_i - 1, \\ b_j = a_j + 1, \\ b_k = a_k, \ \text{if } k \neq i, j \end{cases} \qquad \begin{cases} c_i = a_i + 1, \\ c_j = a_j - 1, \\ c_k = a_k, \ \text{if } k \neq i, j, \end{cases}$$

we can prove that either $f(a_1, \dots, a_{n-2}) < f(b_1, \dots, b_{n-2})$ or $f(a_1, \dots, a_{n-2}) < f(c_1, \dots, c_{n-2})$ holds.

Proof. Here we may assume that $i = 1$ and $j = 2$ without loss of generality and then we also assume that $L_1 \not\equiv L_2$. By assumption, we see that (X, L_1) is a scroll over S. Hence there exists an ample vector bundle $\mathcal{E}$ on S of rank $n-1$ such that $X = \mathbb{P}_S(\mathcal{E})$ and $L_1 = H(\mathcal{E})$. Then for every integer p with $2 \leq p \leq n-2$, we get

$$L_p = H(\mathcal{E}) + \pi^*(T_p),$$

where $T_p \in \mathrm{Pic}(S)$. Here we note that

$$K_X = -(n-1)H(\mathcal{E}) + \pi^*(K_S + c_1(\mathcal{E})).$$

So we have

$$\begin{aligned}
(1) \quad f(a_1, \dots, a_{n-2}) &= (K_X + a_1 L_1 + \cdots + a_{n-2} L_{n-2})^2 L_1^{a_1} \cdots L_{n-2}^{a_{n-2}} \\
&= (-H(\mathcal{E}) + \pi^*(K_S + c_1(\mathcal{E}) + a_2 T_2 + \cdots + a_{n-2} T_{n-2}))^2 \\
&\quad \times H(\mathcal{E})^{a_1} (H(\mathcal{E}) + \pi^*(T_2))^{a_2} \cdots (H(\mathcal{E}) + \pi^*(T_{n-2}))^{a_{n-2}} \\
&= H(\mathcal{E})^n - (2K_S + 2c_1(\mathcal{E}) + a_2 T_2 + \cdots + a_{n-2} T_{n-2}) c_1(\mathcal{E}) \\
&\quad + (K_S + c_1(\mathcal{E}) + a_2 T_2 + \cdots + a_{n-2} T_{n-2})^2 \\
&\quad - 2(K_S + c_1(\mathcal{E}) + a_2 T_2 + \cdots + a_{n-2} T_{n-2})(a_2 T_2 + \cdots + a_{n-2} T_{n-2}) \\
&\quad + \sum_{2 \leq i < j} a_i a_j T_i T_j + \sum_{i=2}^{n-2} \binom{a_i}{2} T_i^2.
\end{aligned}$$

Here we set $b_1 = a_1 - 1$, $b_2 = a_2 + 1$ and $b_k = a_k$ $(k \neq 1, 2)$. Then by (1) we have

$$\begin{aligned}
&f(b_1, \ldots, b_{n-2}) \\
&= H(\mathcal{E})^n - (2K_S + 2c_1(\mathcal{E}) + (a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2})c_1(\mathcal{E}) \\
&\quad + (K_S + c_1(\mathcal{E}) + (a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2})^2 \\
&\quad - 2(K_S + c_1(\mathcal{E}) + (a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2})((a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2}) \\
&\quad + (a_2+1)\left(\sum_{j=3}^{n-2} a_j\right) T_2T_j + \sum_{3 \leq i < j} a_ia_jT_iT_j + \binom{a_2+1}{2}T_2^2 + \sum_{i=3}^{n-2}\binom{a_i}{2}T_i^2.
\end{aligned}$$

Here we note that

$$\begin{aligned}
&(2K_S + 2c_1(\mathcal{E}) + (a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2})c_1(\mathcal{E}) \\
&= (2K_S + 2c_1(\mathcal{E}) + a_2T_2 + \cdots + a_{n-2}T_{n-2})c_1(\mathcal{E}) + T_2c_1(\mathcal{E}),
\end{aligned}$$

$$\begin{aligned}
&(K_S + c_1(\mathcal{E}) + (a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2})^2 \\
&= (K_S + c_1(\mathcal{E}) + a_2T_2 + \cdots + a_{n-2}T_{n-2})^2 \\
&\quad + 2(K_S + c_1(\mathcal{E}) + a_2T_2 + \cdots + a_{n-2}T_{n-2})T_2 + T_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&2(K_S + c_1(\mathcal{E}) + (a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2})((a_2+1)T_2 + a_3T_3 + \cdots + a_{n-2}T_{n-2}) \\
&= 2(K_S + c_1(\mathcal{E}) + a_2T_2 + \cdots + a_{n-2}T_{n-2})(a_2T_2 + \cdots + a_{n-2}T_{n-2}) \\
&\quad + 2(K_S + c_1(\mathcal{E}) + 2a_2T_2 + \cdots + 2a_{n-2}T_{n-2})T_2 + 2T_2^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
&f(a_1, \ldots, a_{n-2}) - f(b_1, \ldots, b_{n-2}) \\
&= T_2c_1(\mathcal{E}) + 2(a_2T_2 + \cdots + a_{n-2}T_{n-2})T_2 - \sum_{j=3}^{n-2} a_jT_2T_j - a_2T_2^2 + T_2^2 \\
&= T_2c_1(\mathcal{E}) + (a_2+1)T_2^2 + (a_3T_3 + \cdots + a_{n-2}T_{n-2})T_2 \\
&= (L_2 - L_1)L_1^{a_1}L_2^{a_2+1}L_3^{a_3} \cdots L_{n-2}^{a_{n-2}}.
\end{aligned} \tag{2}$$

In order to prove Lemma 8.2.1, we assume that both $f(a_1, \ldots, a_{n-2}) \geq f(b_1, \ldots, b_{n-2})$ and $f(a_1, \ldots, a_{n-2}) \geq f(c_1, \ldots, c_{n-2})$ hold. Then by (2) we have

$$L_1^{a_1}L_2^{a_2+2}L_3^{a_3} \cdots L_{n-2}^{a_{n-2}} \geq L_1^{a_1+1}L_2^{a_2+1}L_3^{a_3} \cdots L_{n-2}^{a_{n-2}}$$

and

$$L_1^{a_1+2}L_2^{a_2}L_3^{a_3} \cdots L_{n-2}^{a_{n-2}} \geq L_1^{a_1+1}L_2^{a_2+1}L_3^{a_3} \cdots L_{n-2}^{a_{n-2}}.$$

Hence by [1, Proposition 2.5.1 and Corollary 2.5.4], we see that $L_1 \equiv L_2$. But this contradicts the assumption that $L_1 \not\equiv L_2$. Therefore we get the assertion of Lemma 8.2.1. □

Theorem 8.2.4 *Let $(X, L_1, \ldots, L_{n-2})$ be the type* (10) *in* [12, Remark 5.2.4]. *Then there exists an integer i such that*

$$(K_X + L_1 + \cdots + L_{n-2})^2L_1 \cdots L_{n-2} \leq (K_X + (n-2)L_i)^2L_i^{n-2}.$$

Proof. Let $(a_1, \ldots, a_{n-2}) \in A_{n-2}$. Then we will prove the following.

Claim 8.2.2 *If a_i and a_j satisfy $a_i \geq 1$ and $a_j \geq 1$ for some i and j, then either $f(a_1, \ldots, a_{n-2}) \leq f(b_1, \ldots, b_{n-2})$ or $f(a_1, \ldots, a_{n-2}) \leq f(c_1, \ldots, c_{n-2})$ holds, where*

$$\begin{cases} b_i = 0, \\ b_j = a_i + a_j, \\ b_k = a_k, \ \text{if } k \neq i, j, \end{cases} \qquad \begin{cases} c_i = a_i + a_j, \\ c_j = 0, \\ c_k = a_k, \ \text{if } k \neq i, j. \end{cases}$$

Proof. If $L_i \equiv L_j$, then $f(a_1, \ldots, a_{n-2}) = f(b_1, \ldots, b_{n-2})$ and $f(a_1, \ldots, a_{n-2}) = f(c_1, \ldots, c_{n-2})$ hold. So we may assume that $L_i \not\equiv L_j$. Then we apply Lemma 8.2.1, and we see that one of the following holds.

(A) $f(a_1, \ldots, a_{n-2}) < f(\alpha_1, \ldots, \alpha_{n-2})$.

(B) $f(a_1, \ldots, a_{n-2}) < f(\beta_1, \ldots, \beta_{n-2})$.

Here

$$\begin{cases} \alpha_i = a_i - 1, \\ \alpha_j = a_j + 1, \\ \alpha_k = a_k, \ \text{if } k \neq i, j, \end{cases} \qquad \begin{cases} \beta_i = a_i + 1, \\ \beta_j = a_j - 1, \\ \beta_k = a_k, \ \text{if } k \neq i, j. \end{cases}$$

Assume that the case (A) holds. If $\alpha_i = 0$, then this is done. So we may assume that $\alpha_i \geq 1$. Then $\alpha_j \geq 1$ and Lemma 8.2.1 implies that one of the following holds.

(A′) $f(\alpha_1, \ldots, \alpha_{n-2}) < f(\gamma_1, \ldots, \gamma_{n-2})$.

(B′) $f(\alpha_1, \ldots, \alpha_{n-2}) < f(\delta_1, \ldots, \delta_{n-2})$.

Here

$$\begin{cases} \gamma_i = \alpha_i - 1, \\ \gamma_j = \alpha_j + 1, \\ \gamma_k = \alpha_k, \ \text{if } k \neq i, j, \end{cases} \qquad \begin{cases} \delta_i = \alpha_i + 1, \\ \delta_j = \alpha_j - 1, \\ \delta_k = \alpha_k, \ \text{if } k \neq i, j. \end{cases}$$

But since $(\delta_1, \ldots, \delta_{n-2}) = (a_1, \ldots, a_{n-2})$, the case (B′) cannot occur because $f(a_1, \ldots, a_{n-2}) < f(\delta_1, \ldots, \delta_{n-2})$ from (A) and (B′). By repeating this process, we find that $f(a_1, \ldots, a_{n-2}) < f(b_1, \ldots, b_{n-2})$ holds.

If we assume that the case (B) holds, then by the same argument as above $f(a_1, \ldots, a_{n-2}) < f(c_1, \ldots, c_{n-2})$ holds. So we get the assertion of Claim 8.2.2. □

We go back to the proof of Theorem 8.2.4. By using Claim 8.2.2 repeatedly, there exist $(d_1, \ldots, d_{n-2}) \in A_{n-2}$ and an integer i such that $d_i = n-2$ and $d_j = 0$ for every j with $j \neq i$ and $f(a_1, \ldots, a_{n-2}) \leq f(d_1, \ldots, d_{n-2})$. Therefore we get the assertion of Theorem 8.2.4. □

Theorem 8.2.5 *Let $(X, L_1, \ldots, L_{n-2})$ be the type* (10) *in* [12, Remark 5.2.4]. *Then the following inequality holds.*

$$(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \begin{cases} \leq 8\chi_2^H(X, L_1, \ldots, L_{n-2}) & \text{if } \kappa(S) \neq 2, \\ < 9\chi_2^H(X, L_1, \ldots, L_{n-2}) & \text{if } \kappa(S) = 2. \end{cases}$$

Proof. By Theorem 8.2.4, there exists an integer i such that

$$(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \leq (K_X + (n-2)L_i)^2 L_i^{n-2}.$$

On the other hand, by [9, Theorem 3.1.1 (4)], we have

$$(K_X + (n-2)L_i)^2 L_i^{n-2} \begin{cases} \leq 8\chi_2^H(X, L_i) & \text{if } \kappa(S) \neq 2, \\ < 9\chi_2^H(X, L_i) & \text{if } \kappa(S) = 2. \end{cases}$$

Here we note that $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S)$, $g_2(X, L_i) = h^2(\mathcal{O}_X) = h^2(\mathcal{O}_S)$ and $g_2(X, L_1, \dots, L_{n-2}) = h^2(\mathcal{O}_X) = h^2(\mathcal{O}_S)$ by [11, Example 2.1 (H)]. Hence $\chi_2^H(X, L_i) = 1 - h^1(\mathcal{O}_X) + g_2(X, L_i) = \chi(\mathcal{O}_S) = 1 - h^1(\mathcal{O}_X) + g_2(X, L_1, \dots, L_{n-2}) = \chi_2^H(X, L_1, \dots, L_{n-2})$, and we get the assertion of Theorem 8.2.5. □

Finally we are going to investigate (5) in Conjecture 8.2.1.

Theorem 8.2.6 *Let $(X, L_1, \dots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $n-2$ with $n \geq 4$. Assume that $\kappa(X) \geq 0$. Then*

$$12\chi_2^H(X, L_1, \dots, L_{n-2}) > (K_X + L_1 + \dots + L_{n-2})^2 L_1 \cdots L_{n-2}.$$

Proof. By taking a reduction of $(X, L_1, \dots, L_{n-2})$, we may assume that $K_X + L_1 + \dots + L_{n-2}$ is nef and $(n-2)$-big by [12, Remark 5.2.4] because $\kappa(X) \geq 0$. Hence by using Theorem 7.1 (setting $B := (L_1 + \dots + L_{n-2})/n$), we see that

$$\begin{aligned}
&\chi_2^H(X, L_1, \dots, L_{n-2}) \\
&= 1 - h^1(\mathcal{O}_X) + g_2(X, L_1, \dots, L_{n-2}) \\
&= \frac{1}{6}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} + \frac{1}{4}\left(\sum_{1 \leq j < k \leq n-2} L_j L_k\right) L_1 \cdots L_{n-2} \\
&\quad + \frac{1}{4} K_X \left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} + \frac{1}{12}(c_2(X) + K_X^2) L_1 \cdots L_{n-2} \\
&\geq \frac{1}{6}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} + \frac{1}{4}\left(\sum_{1 \leq j < k \leq n-2} L_j L_k\right) L_1 \cdots L_{n-2} \\
&\quad + \frac{1}{4} K_X \left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} + \frac{1}{12} K_X^2 L_1 \cdots L_{n-2} \\
&\quad - \frac{n-1}{12n} K_X \left(\sum_{j=1}^{n-2} L_j\right) L_1 \cdots L_{n-2} - \frac{n-1}{24n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} \\
&= \frac{1}{12}(K_X + L_1 + \dots + L_{n-2})^2 L_1 \cdots L_{n-2} \\
&\quad + \frac{1}{12n} K_X (L_1 + \dots + L_{n-2}) L_1 \cdots L_{n-2} + \frac{1}{24n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} \\
&\quad + \frac{1}{24}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\
&> \frac{1}{12}(K_X + L_1 + \dots + L_{n-2})^2 L_1 \cdots L_{n-2}.
\end{aligned}$$

Here we note that $K_X(L_1 + \dots + L_{n-2}) L_1 \cdots L_{n-2} \geq 0$ because $\kappa(X) \geq 0$. So we get the assertion. □

9 Applications

In this section, we will provide two applications.

9.1 The sectional geometric genus of complete intersections of hypersurfaces in $\mathbb{P}^N$

By using the notion of the sectional geometric genus of multi-polarized manifolds, we calculate the sectional geometric genus of (X, L), where X is a complete intersection of hypersurfaces in $\mathbb{P}^N$ and $L := \mathcal{O}_{\mathbb{P}^N}(1)|_X$.

Theorem 9.1.1 *Let X be a projective variety such that X is a complete intersection of hypersurfaces D_j of $\mathbb{P}^N$ with $D_j \in |\mathcal{O}_{\mathbb{P}^N}(d_j)|$ for any j with $1 \leq j \leq r$. Let $n := \dim X = N - r$ and $L := \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Then for every integer i with $0 \leq i \leq n-1 = N-r-1$*

$$g_i(X, L) = \sum_{u=1}^{r}(-1)^{r-u} \sum_{(p_1,\ldots,p_r)\in S(r)_u} \binom{d_1p_1 + \cdots + d_rp_r - 1}{r+i}.$$

Here

$$S(r)_u = \{(p_1, \cdots, p_r) \mid p_m \in \mathbb{Z},\ 0 \leq p_m \leq 1,\ \sharp\{m \mid p_m = 1\} = u\}.$$

Proof. Here we note that X and D_j are not smooth in general. First we will prove the following:

Claim 9.1.1 $g_i(X, L) = g_i(\mathbb{P}^N, D_1, \ldots, D_r, \underbrace{H, \ldots, H}_{N-i-r})$*, where $H := \mathcal{O}_{\mathbb{P}^N}(1)$.*

Proof. By the Bertini theorem we can take a general member $D'_j \in |\mathcal{O}_{\mathbb{P}^N}(d_j)|$ for any j such that the following holds: $X'_k := D'_1 \cap \ldots \cap D'_k$ is a *smooth* projective variety of dimension $N-k$ for every k with $1 \leq k \leq r$. Set $X' := D'_1 \cap \ldots \cap D'_r$.

Then by using [11, Theorem 2.3] we have

$$g_i(X', H|_{X'}) = g_i(\mathbb{P}^N, D'_1, \ldots, D'_r, H, \ldots, H).$$

Let $N_{X'/\mathbb{P}^N}$ (resp. $N_{X/\mathbb{P}^N}$) be the normal bundle to X' (resp. X) in $\mathbb{P}^N$. Then $c(N_{X'/\mathbb{P}^N}) = \prod_{k=1}^{r}(1 + d_kH|_{X'})$ and $c(N_{X/\mathbb{P}^N}) = \prod_{k=1}^{r}(1 + d_kH|_X)$. Since

$$c(T_{X'}) = (1 + H|_{X'})^{N+1} / \prod_{k=1}^{r}(1 + d_kH|_{X'})$$

(see [14, Example 3.2.12]), by [7, Theorem 2.1] we get

$$g_i(X', H|_{X'}) = a(H|_{X'})^{N-r} + (-1)^{i+1}\left(\chi(\mathcal{O}_{X'}) - \sum_{k=0}^{n-i}(-1)^{n-k}h^{n-k}(\mathcal{O}_{X'})\right),$$

where $a \in \mathbb{Q}$. On the other hand since

$$c(T_X) = (1 + H|_X)^{N+1} / \prod_{k=1}^{r}(1 + d_kH|_X),$$

by [7, Theorem 2.1] we have

$$g_i(X, H|_X) = a(H|_X)^{N-r} + (-1)^{i+1}\left(\chi(\mathcal{O}_X) - \sum_{k=0}^{n-i}(-1)^{n-k}h^{n-k}(\mathcal{O}_X)\right).$$

Here we note that $h^0(\mathcal{O}_X)=h^0(\mathcal{O}_{X'})=1$ and $h^j(\mathcal{O}_X)=h^j(\mathcal{O}_{X'})=0$ for every j with $0<j<n$ (see also [15, Chapter I, Section 3, Theorem 3.4 (a) and Chapter III, Section 5, Exercise 5.5 (c)]). Since

$$\chi(\mathcal{O}_{X'})-\sum_{k=0}^{n-i}(-1)^{n-k}h^{n-k}(\mathcal{O}_{X'})=\chi(\mathcal{O}_X)-\sum_{k=0}^{n-i}(-1)^{n-k}h^{n-k}(\mathcal{O}_X),\quad (H|_{X'})^{N-r}=(H|_X)^{N-r}$$

and D_k is linearly equivalent to D'_k, we see that

$$\begin{aligned} g_i(X,L) &= g_i(X,H|_X)\\ &= g_i(X',H|_{X'})\\ &= g_i(\mathbb{P}^N,D'_1,\dots,D'_r,H,\dots,H)\\ &= g_i(\mathbb{P}^N,D_1,\dots,D_r,H,\dots,H). \end{aligned}$$

This completes the proof of Claim 9.1.1. □

Next by using [11, Corollary 2.3], we calculate $g_i(\mathbb{P}^N,D_1,\dots,D_r,\underbrace{H,\dots,H}_{N-i-r})$. Here we note that

$$\sum_{j=0}^{N-i}(-1)^{N-i-j}h^{N-j}(\mathcal{O}_{\mathbb{P}^N})=0$$

for any $i\geq 1$. First we consider the case where $N-i-r=1$. Then $i=N-r-1=n-1$ and by [11, Corollary 2.3] we have

$$\begin{aligned} &g_{n-1}(\mathbb{P}^N,D_1,\dots,D_r,H)\\ &=\sum_{u=1}^{r+1}\left\{(-1)^{r+1-u}\sum_{(p_1,\dots,p_r,q)\in S(r+1)_u}h^0(K_{\mathbb{P}^N}+p_1D_1+\cdots+p_rD_r+qH)\right\}\\ &=\sum_{u=1}^{r+1}(-1)^{r+1-u}\sum_{(p_1,\dots,p_r,q)\in S(r+1)_u}\binom{d_1p_1+\cdots+d_rp_r+q-1}{N}\\ &=\sum_{u=1}^{r+1}(-1)^{r+1-u}\left\{\sum_{(p_1,\dots,p_r)\in S(r)_{u-1}}\binom{d_1p_1+\cdots+d_rp_r}{N}\right\}\\ &\quad+\sum_{u=1}^{r}(-1)^{r+1-u}\sum_{(p_1,\dots,p_r)\in S(r)_u}\binom{d_1p_1+\cdots+d_rp_r-1}{N}\\ &=\sum_{u=0}^{r}(-1)^{r-u}\sum_{(p_1,\dots,p_r)\in S(r)_u}\binom{d_1p_1+\cdots+d_rp_r}{N}\\ &\quad-\sum_{u=1}^{r}(-1)^{r-u}\sum_{(p_1,\dots,p_r)\in S(r)_u}\binom{d_1p_1+\cdots+d_rp_r-1}{N}\\ &=\sum_{u=1}^{r}(-1)^{r-u}\sum_{(p_1,\dots,p_r)\in S(r)_u}\left\{\binom{d_1p_1+\cdots+d_rp_r}{N}-\binom{d_1p_1+\cdots+d_rp_r-1}{N}\right\}. \end{aligned}$$

(Here we note that if $u=0$, then $(-1)^r\sum_{(p_1,\cdots,p_r)\in S(r)_0}\binom{d_1p_1+\cdots+d_rp_r}{N}=0$.) Hence

$$g_{n-1}(\mathbb{P}^N,D_1,\dots,D_r,H)$$

$$= \sum_{u=1}^{r}(-1)^{r-u} \sum_{(p_1,\ldots,p_r)\in S(r)_u} \left\{ \binom{d_1p_1+\cdots+d_rp_r}{N} - \binom{d_1p_1+\cdots+d_rp_r-1}{N} \right\}$$
$$= \sum_{u=1}^{r}(-1)^{r-u} \sum_{(p_1,\ldots,p_r)\in S(r)_u} \binom{d_1p_1+\cdots+d_rp_r-1}{N-1}$$
$$= \sum_{u=1}^{r}(-1)^{r-u} \sum_{(p_1,\ldots,p_r)\in S(r)_u} \binom{d_1p_1+\cdots+d_rp_r-1}{r+n-1}.$$

Next we consider the case where $N-i-r\geq 2$ and we calculate $g_i(\mathbb{P}^N, D_1,\ldots,D_r,\underbrace{H,\ldots,H}_{N-i-r})$.

Here we note that by [11, Theorem 2.3]

$$\begin{aligned} & g_i(\mathbb{P}^N, D_1,\ldots,D_r,\underbrace{H,\ldots,H}_{N-i-r}) \\ & = g_i(\mathbb{P}^{r+i+1}, \mathcal{O}_{\mathbb{P}^{r+i+1}}(d_1),\ldots,\mathcal{O}_{\mathbb{P}^{r+i+1}}(d_r),\mathcal{O}_{\mathbb{P}^{r+i+1}}(1)). \end{aligned}$$

On the other hand, by above

$$\begin{aligned} & g_i(\mathbb{P}^{r+i+1}, \mathcal{O}_{\mathbb{P}^{r+i+1}}(d_1),\ldots,\mathcal{O}_{\mathbb{P}^{r+i+1}}(d_r),\mathcal{O}_{\mathbb{P}^{r+i+1}}(1)) \\ & = \sum_{u=1}^{r}(-1)^{r-u} \sum_{(p_1,\ldots,p_r)\in S(r)_u} \binom{d_1p_1+\cdots+d_rp_r-1}{r+i}. \end{aligned}$$

Hence

$$\begin{aligned} & g_i(\mathbb{P}^N, D_1,\ldots,D_r,H,\ldots,H) \\ & = \sum_{u=1}^{r}(-1)^{r-u} \sum_{(p_1,\ldots,p_r)\in S(r)_u} \binom{d_1p_1+\cdots+d_rp_r-1}{r+i}. \end{aligned}$$

So we get the assertion. □

9.2 The sectional m-genus of multi-polarized manifolds

Here we define the ith sectional m-genus of multi-quasi-polarized manifolds.

Definition 9.2.1 Let m be an integer with $m\geq 2$. Let $(X,L_1,\cdots,L_{n-i})$ be an n-dimensional multi-quasi-polarized manifold of type $n-i$, where i is an integer with $0\leq i\leq n-1$. Let $\mathcal{F}$ be a coherent sheaf on X.
(1) Assume that $K_X+L_1+\cdots+L_{n-i}$ is nef and $(n-i)$-big. Then the *ith sectional m-genus* $p_m^i(X,L_1,\cdots,L_{n-i};\mathcal{F})$ is defined by the following:

$$\begin{aligned} & p_m^i(X,L_1,\cdots,L_{n-i};\mathcal{F}) \\ & := \begin{cases} g_{i-1}(X,(m-1)(K_X+L_1+\cdots+L_{n-i}),L_1,\ldots,L_{n-i};\mathcal{F}) \\ \quad +g_i(X,L_1,\ldots,L_{n-i};\mathcal{F})-h^{i-1}(\mathcal{F}) \ \text{ if } 1\leq i\leq n-1, \\ g_0(X,L_1,\ldots,L_n;\mathcal{F}) \quad \text{if } i=0. \end{cases} \end{aligned}$$

(2) If $\mathcal{F}=\mathcal{O}_X$, then we set $p_m^i(X,L_1,\cdots,L_{n-i}):=p_m^i(X,L_1,\cdots,L_{n-i};\mathcal{O}_X)$.
(3) If $\mathcal{F}=\mathcal{O}_X$ and $L=L_1=\cdots=L_{n-i}$, then we set $p_m^i(X,L):=p_m^i(X,L,\ldots,L;\mathcal{O}_X)$.

Theorem 9.2.1 *Let m, n, and i be integers with $m\geq 2$, $n\geq 2$ and $1\leq i\leq n-1$. Let $(X,L_1,\ldots,L_{n-i})$ be an n-dimensional multi-quasi-polarized manifold of type $(n-i)$. Assume that*

$n \geq 2$, $K_X + L_1 + \cdots + L_{n-i}$ is nef and $(n-i)$-big and $|L_j|$ is base point free for every j with $1 \leq j \leq n-i$. Here we use notation in Notation 7.1. Then

$$p_m^i(X, L_1, \dots, L_{n-i}) = h^0(mK_{X_{n-i}}).$$

Proof. By definition and the proof of [11, Theorem 2.3] we get

$$\begin{aligned}
&p_m^i(X, L_1, \dots, L_{n-i}) \\
&= g_{i-1}(X, (m-1)(K_X + L_1 + \cdots + L_{n-i}), L_1, \dots, L_{n-i}) \\
&\quad + g_i(X, L_1, \dots, L_{n-i}) - h^{i-1}(\mathcal{O}_X) \\
&= g_{i-1}(X_{n-i}, (m-1)K_{X_{n-i}}) + h^i(\mathcal{O}_{X_{n-i}}) - h^{i-1}(\mathcal{O}_{X_{n-i}}).
\end{aligned}$$

Since $K_X + L_1 + \cdots + L_{n-i}$ is nef and $(n-i)$-big, we see that $K_{X_{n-i}}$ is nef and big because $(K_X + L_1 + \cdots + L_{n-i})^i L_1 \cdots L_{n-i} > 0$ by [1, Lemma 2.5.8]. Hence by [5, Theorem 2.3] we have

$$\begin{aligned}
g_{i-1}(X_{n-i}, (m-1)K_{X_{n-i}}) &= h^0(K_{X_{n-i}} + (m-1)K_{X_{n-i}}) - h^i(\mathcal{O}_{X_{n-i}}) + h^{i-1}(\mathcal{O}_{X_{n-i}}) \\
&= h^0(mK_{X_{n-i}}) - h^i(\mathcal{O}_{X_{n-i}}) + h^{i-1}(\mathcal{O}_{X_{n-i}}).
\end{aligned}$$

Therefore we get the assertion. □

Theorem 9.2.2 *Let m be an integer with $m \geq 2$. Let $(X, L_1, \dots, L_{n-1})$ be an n-dimensional multi-quasi-polarized manifold of type $(n-1)$. Assume that $n \geq 2$ and $K_X + L_1 + \cdots + L_{n-1}$ is nef and $(n-1)$-big. Then*

$$p_m^1(X, L_1, \dots, L_{n-1}) \geq 2m - 1.$$

Proof. By definition we get

$$\begin{aligned}
p_m^1(X, L_1, \dots, L_{n-1}) &= g_0(X, (m-1)(K_X + L_1 + \cdots + L_{n-1}), L_1, \dots, L_{n-1}) \\
&\quad + g_1(X, L_1, \dots, L_{n-1}) - h^0(\mathcal{O}_X) \\
&= \frac{2m-1}{2}(K_X + L_1 + \cdots + L_{n-1})L_1 \cdots L_{n-1}.
\end{aligned}$$

On the other hand, since $K_X + L_1 + \cdots + L_{n-1}$ is nef and $(n-1)$-big, we see that $(K_X + L_1 + \cdots + L_{n-1})L_1 \cdots L_{n-1} > 0$ by [1, Lemma 2.5.8]. Moreover $(K_X + L_1 + \cdots + L_{n-1})L_1 \cdots L_{n-1}$ is even. Hence we get the assertion. □

Remark 9.2.1 If $\dim X = 1$, $m \geq 2$ and K_X is nef and big, then by the Riemann-Roch theorem

$$h^0(mK_X) = (2m-1)(h^1(\mathcal{O}_X) - 1).$$

Since K_X is nef and big, we have $h^1(\mathcal{O}_X) \geq 2$. Hence we get $h^0(mK_X) \geq 2m-1$. So Theorem 9.2.2 can be regarded as a generalization of this result.

Theorem 9.2.3 *Let (X, L) be a polarized manifold of dimension $n \geq 3$, and let m be an integer with $m \geq 2$. Assume that $K_X + (n-1)L$ is nef and $(n-1)$-big. If $p_m^1(X, L) = 2m-1$, then (X, L) is one of the following types.*

(1) *$K_X + (n-3)L = \mathcal{O}_X$.*

(2) *X is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree 6 and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.*

(3) *(X, L) is a simple blowing up of the type (2) above. In this case $n = 3$.*

(4) (X, L) *is a scroll over a smooth surface, and one of the types in* [3, (3.4) Theorem].

(5) (X, L) *is a quadric fibration over a smooth curve, and one of the types in* [2, (3.7) and (3.30) Theorem].

Proof. By the proof of Theorem 9.2.2, we see that $(K_X+(n-1)L)L^{n-1} = 2$. Hence $g_1(X, L) = 2$. So by the classification of (X, L) with $g_1(X, L) = 2$ (see [2, (1.10) Theorem]), (X, L) is one of the above types because $K_X + (n-1)L$ is nef and $(n-1)$-big. □

Theorem 9.2.4 *Let n be an integer with $n \geq 3$. Let $(X, L_1, \ldots, L_{n-2})$ be an n-dimensional multi-polarized manifold of type $n-2$. Assume that one of the following conditions holds:*

(a) *$n = 3$ and $K_X + L_1$ is nef and 1-big.*

(b) *$n \geq 4$, $\kappa(X) \geq 0$ and $K_X + L_1 + \cdots + L_{n-2}$ is nef.*

Then for any integer m with $m \geq 2$ we have

$$p_m^2(X, L_1, \ldots, L_{n-2}) \geq 1 + \frac{m(m-1)}{2}.$$

Proof. By definition we get

$$\begin{aligned} p_m^2(X, L_1, \ldots, L_{n-2}) &= g_1(X, (m-1)(K_X + L_1 + \cdots + L_{n-2}), L_1, \ldots, L_{n-2}) \\ &\quad + g_2(X, L_1, \ldots, L_{n-2}) - h^1(\mathcal{O}_X) \\ &= 1 + \frac{m(m-1)}{2}(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} \\ &\quad + \chi_2^H(X, L_1, \ldots, L_{n-2}) - 1. \end{aligned}$$

On the other hand by assumption we have $(K_X + L_1 + \cdots + L_{n-2})^2 L_1 \cdots L_{n-2} > 0$ by [1, Lemma 2.5.8]. Moreover $\chi_2^H(X, L_1, \ldots, L_{n-2}) \geq 1$ by [9, Theorem 3.3.1(2)] and Theorem 8.2.2. Hence we get the assertion. □

Remark 9.2.2 If $\dim X = 2$, $m \geq 2$ and K_X is nef and big, then by the Riemann-Roch theorem and the Kawamata-Viehweg vanishing theorem we have

$$h^0(mK_X) = \frac{m(m-1)}{2} K_X^2 + \chi(\mathcal{O}_X).$$

Since K_X is nef and big, we have $\chi(\mathcal{O}_X) \geq 1$. Hence we get $h^0(mK_X) \geq 1 + \frac{m(m-1)}{2}$. Therefore Theorem 9.2.4 can be regarded as a generalization of this result.

Theorem 9.2.5 *Let (X, L) be a polarized manifold of dimension 3, and let m be an integer with $m \geq 2$. Assume that $K_X + L$ is nef and 1-big. Then $p_m^2(X, L) = 1 + (m(m-1)/2)$ if and only if $\mathcal{O}(K_X) = \mathcal{O}_X$, $h^1(\mathcal{O}_X) = 0$, $h^0(L) = 1$ and $L^3 = 1$.*

Proof. First we are going to prove the "only if" part. By the proof of Theorem 9.2.4, we see that $(K_X + L)^2 L = 1$.

Assume that $(K_X + L)^3 > 0$. Then by Proposition 7.1, we have $(K_X + L)L^2 = 1$. Again by using Proposition 7.1, we also have $L^3 = 1$. Hence $(K_X + 2L)L^2 = 2$. Therefore $g_1(X, L) = 2$. By the classification of (X, L) with $g_1(X, L) = 2$ (see [2, (1.10) Theorem]) we see that $\mathcal{O}(K_X) = \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$ because $K_X + L$ is nef and 1-big. Moreover since $p_m^2(X, L) = 1 + (m(m-1)/2)$, we see from the proof of Theorem 9.2.4 that $\chi_2^H(X, L) = 1$. Hence $g_2(X, L) = 0$ because $h^1(\mathcal{O}_X) = 0$. Therefore by [11, Corollary 2.3] or [5, Theorem 2.3] we have $h^0(L) = h^0(K_X + L) = 1$ since $h^3(\mathcal{O}_X) = 1$ and $h^2(\mathcal{O}_X) = h^1(K_X) = h^1(\mathcal{O}_X) = 0$.

Assume that $(K_X+L)^3=0$. Then (X,L) is a quadric fibration over a normal surface S. Then there exists a surjective morphism $f: X \to S$ with connected fibers such that $K_X+L=f^*(A)$ for some ample line bundle A on S. Since $(K_X+L)^2L=1$, we get $(f^*(A))^2L=1$. But by Proposition 7.2, this is impossible in this case.

Next we prove the "if" part. If $\mathcal{O}(K_X)=\mathcal{O}_X$, $h^1(\mathcal{O}_X)=0$, $h^0(L)=1$ and $L^3=1$, then easy calculations show that $p_m^2(X,L)=1+(m(m-1)/2)$ holds. □

Remark 9.2.3 There exists an example of a polarized 3-fold (X,L) with $\mathcal{O}(K_X)=\mathcal{O}_X$, $h^1(\mathcal{O}_X)=0$, $h^0(L)=1$ and $L^3=1$. (See [2, (2.7) and Remark (2.13)].)

Furthermore we can prove the following.

Theorem 9.2.6 *Let m, n and i be integers such that $m\geq 2$, $n\geq 2$ and $1\leq i\leq n-1$. Let $(X,L_1,\dots,L_{n-i})$ be an n-dimensional multi-quasi-polarized manifold of type $n-i$. Assume that $K_X+L_1+\cdots+L_{n-i}$ is nef and $(n-i)$-big, and $\mathrm{Bs}|L_j|=\emptyset$ for every integer j with $1\leq j\leq n-i$. Then*

$$p_m^i(X,L_1,\dots,L_{n-i})\geq mg_i(X,L_1,\dots,L_{n-i})-(m-1).$$

Proof. Here we note that $g_i(X,L_1,\dots,L_{n-i})\geq 0$ by [12, Theorem 4.1]. Since

$$p_m^i(X,L_1,\dots,L_{n-i})\geq 0$$

by Theorem 9.2.1, the assertion is true if $g_i(X,L_1,\dots,L_{n-i})=0$. So we may assume that $g_i(X,L_1,\dots,L_{n-i})\geq 1$. Here we use Notation 7.1. Then we note that

$$g_i(X,L_1,\dots,L_{n-i})=h^0(K_{X_{n-i}}) \quad\text{and}\quad p_m^i(X,L_1,\dots,L_{n-i})=h^0(mK_{X_{n-i}}).$$

Now we are going to prove the assertion by induction on m. First we consider the case where $m=2$. Since $h^0(K_{X_{n-i}})=g_i(X,L_1,\dots,L_{n-i})\geq 1$ by assumption, we have $h^0(2K_{X_{n-i}})\geq 2h^0(K_{X_{n-i}})-1$ by Lemma 7.1. Therefore we get the assertion for $m=2$.

Next we assume that the assertion is true for $m=k$ with $k\geq 2$. Then

$$p_k^i(X,L_1,\dots,L_{n-i})\geq kg_i(X,L_1,\dots,L_{n-i})-(k-1)$$

holds by assumption, and $g_i(X,L_1,\dots,L_{n-i})\geq 1$ implies that $h^0(kK_{X_{n-i}})=p_k^i(X,L_1,\dots,L_{n-i})\geq 1$. Since $h^0(K_{X_{n-i}})=g_i(X,L_1,\dots,L_{n-i})\geq 1$, by Lemma 7.1 we obtain

$$\begin{aligned}
p_{k+1}^i(X,L_1,\dots,L_{n-i}) &= h^0((k+1)K_{X_{n-i}})\\
&\geq h^0(kK_{X_{n-i}})+h^0(K_{X_{n-i}})-1\\
&= p_k^i(X,L_1,\dots,L_{n-i})+g_i(X,L_1,\dots,L_{n-i})-1\\
&\geq kg_i(X,L_1,\dots,L_{n-i})-(k-1)+g_i(X,L_1,\dots,L_{n-i})-1\\
&= (k+1)g_i(X,L_1,\dots,L_{n-i})-k.
\end{aligned}$$

So we get the assertion for $m=k+1$, and this completes the proof. □

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