Invariants of ample vector bundles on smooth projective varieties *†‡

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Abstract

Let X be a smooth projective variety of dimension n, let $\mathcal{E}$ be an ample vector bundle of rank r on X with $1 \leq r \leq n$. Then we are going to introduce some invariants of $(X, \mathcal{E})$ which are considered to be a generalization of invariants of polarized manifolds we introduced before. Moreover we will study some properties of these and some relationships between these.

1 Introduction

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then the pair (X, L) is called a *polarized variety*. Moreover if X is smooth, then (X, L) is called a polarized *manifold*.

When we study polarized varieties, it is useful to use their invariants. The sectional genus $g(X, L)$ of (X, L) is one of the well-known invariants of (X, L). In [3] (resp. [5]) we defined the notion of the *ith sectional geometric genus* $g_i(X, L)$ (resp. *the ith sectional H-arithmetic genus* $\chi_i^H(X, L)$) *of* (X, L) for every integer i with $0 \leq i \leq n$. Here we explain the meaning of these invariants if X is smooth, L is base point free and i is an integer with $1 \leq i \leq n-1$. Let $H_1, \ldots, H_{n-i}$ be general members of $|L|$. We put $X_{n-i} := H_1 \cap \ldots \cap H_{n-i}$. Then X_{n-i} is smooth with $\dim X_{n-i} = i$, and we can show that $g_i(X, L) = h^i(\mathcal{O}_{X_{n-i}})$ and $\chi_i^H(X, L) = \chi(\mathcal{O}_{X_{n-i}})$. (Here we call $\chi(\mathcal{O}_Y)$ the H-arithmetic genus of a projective variety Y (see also [5, Definition 1.5]).)

These induce the notion of the *ith sectional invariant of* (X, L) *associated with an invariant.*

Definition 1.1 Let (X, L) be a polarized manifold of dimension n. Let $I(Y)$ (or I) be an invariant of a smooth projective variety Y of dimension i, where i is an integer with $0 \leq i \leq n$. Then an invariant $F_i(X, L)$ of (X, L) is called the *ith sectional invariant of* (X, L) *associated with the invariant* I if $F_i(X, L) = I(X_{n-i})$ under the assumption that $\mathrm{Bs}|L| = \emptyset$.

The ith sectional geometric genus (resp. the ith sectional H-arithmetic genus) is the ith sectional invariant of (X, L) associated with the geometric genus (resp. the H-arithmetic genus). By the definition of the ith sectional invariants, the ith sectional invariants are expected to reflect properties of i-dimensional geometry. So we can expect that we are able to find interesting properties of (X, L) by using its ith sectional invariants.

In [6], we defined other ith sectional invariants, that is, the *ith sectional Euler number* $e_i(X, L)$, the *ith sectional Betti number* $b_i(X, L)$, and the *ith sectional Hodge number* $h_i^{j,i-j}(X, L)$ *of type*

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$(j, i-j)$ *of* (X, L) (see Definition 2.2.1 below) and we studied some properties of these. The meaning of these invariants is the following. Assume that X is smooth, L is base point free and i is an integer with $1 \leq i \leq n-1$. Let $H_1, \ldots, H_{n-i}$ be general members of $|L|$. We put $X_{n-i} := H_1 \cap \ldots \cap H_{n-i}$. Then X_{n-i} is smooth with $\dim X_{n-i} = i$, and we see that $e_i(X, L) = e(X_{n-i})$, $b_i(X, L) = h^i(X_{n-i}, \mathbb{C})$ and $h_i^{j,i-j}(X, L) = h^{j,i-j}(X_{n-i})$.

The main purpose of this paper is to define a vector bundle version of these invariants as a generalization, and to give a frame of investigation of generalized polarized manifolds by using sectional invariants defined in this paper. In future, we will give detailed investigations such as classification of (multi-)generalized polarized manifolds by their sectional invariants.

Let X be a smooth projective variety with $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle on X with rank $\mathcal{E} = r$. We assume that $r \leq n$. In Section 3, we will define the *c_r-sectional H-arithmetic genus* $\chi^H_{n,r}(X, \mathcal{E})$, the *$c_r$-sectional geometric genus* $g_{n,r}(X, \mathcal{E})$, the *c_r-sectional Euler number* $e_{n,r}(X, \mathcal{E})$, the *c_r-sectional Betti number* $b_{n,r}(X, \mathcal{E})$ and the *c_r-sectional Hodge number* $h^{j,n-r-j}_{n,r}(X, \mathcal{E})$ *of type* $(j, n-r-j)$ of $(X, \mathcal{E})$.

Moreover in Section 4 we will study fundamental properties of these, which will be useful for investigations by these invariants. In Section 5, as a special case, we consider the case where $\mathcal{E}$ is a direct sum of ample line bundles. In 5.1, we will define the c_r-sectional invariants of multi-polarized manifolds (see Definition 5.1.1). In 5.2, we will show that some invariants defined before are special cases of these invariants (see Propositions 5.2.1 and 5.2.2). In 5.3, we will study some properties of the sectional Euler numbers, the sectional Betti numbers and the sectional Hodge numbers of multi-polarized manifolds. In Section 6, we will propose some problems and conjectures.

Finally we note that in a forthcoming paper [10], we will make a study of (multi-)polarized manifolds by using invariants which will be defined in this paper.

2 Preliminaries

In this section, let X be a smooth projective variety of dimension n unless otherwise mensioned.

2.1 Some notation

In 2.1, we will give some notation which will be used later.

Definition 2.1.1 Let $\mathcal{F}$ be a vector bundle on X. Then for every integer j with $j \geq 0$, the *jth Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$* is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\geq 0} s_j(\mathcal{F})t^j$.

Remark 2.1.1 (a) Let $\mathcal{F}$ be a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the jth Segre class which is defined in [12, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.
(b) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

Definition 2.1.2 Let $\mathcal{E}$ (resp. L) be an ample vector bundle (resp. an ample line bundle) on X. Then the pair $(X, \mathcal{E})$ (resp. (X, L)) is called a *generalized polarized manifold* (resp. *polarized manifold*).

Definition 2.1.3 Let $(X, \mathcal{E})$ be an n-dimensional generalized polarized manifold with rank $\mathcal{E} = r$. We assume that $r \leq n$. For every integer p with $0 \leq p \leq n-r$ we set

$$C^{n,r}_p(X, \mathcal{E}) := \sum_{k=0}^{p} c_k(X)s_{p-k}(\mathcal{E}^\vee).$$

Remark 2.1.2 Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension n. Let r be the rank of $\mathcal{E}$ with $r\leq n-1$. Assume that there exists a section of $H^0(\mathcal{E})$ whose zero locus Z is smooth and $\dim Z = n-r$. Then there is an exact sequence $0\to \mathcal{T}_Z\to \mathcal{T}_X|_Z\to \mathcal{E}|_Z\to 0$ and we have $c_t(\mathcal{T}_X|_Z)=c_t(\mathcal{T}_Z)c_t(\mathcal{E}|_Z)$, where $\mathcal{T}_X$ (resp. $\mathcal{T}_Z$) is the tangent bundle of X (resp. Z). Hence we have $c_t(Z)=c_t((\mathcal{T}_X)_Z)c_t(\mathcal{E}|_Z)^{-1}=c_t((\mathcal{T}_X)_Z)s_t(\mathcal{E}^\vee|_Z)$. Therefore we get

$$c_i(Z)=\sum_{j=0}^{i}c_j(X)s_{i-j}(\mathcal{E}^\vee)|_Z=\sum_{j=0}^{i}c_j(X)s_{i-j}(\mathcal{E}^\vee)c_r(\mathcal{E})=C_i^{n,r}(X,\mathcal{E})c_r(\mathcal{E}).$$

In particular, we have $K_Z=(K_X+c_1(\mathcal{E}))c_r(\mathcal{E})$.

2.2 Sectional invariants of polarized manifolds

In 2.2, we will review the sectional invariants of polarized manifolds.

Notation 2.2.1 (1) Let (X,L) be a polarized manifold of dimension n. Then the Euler-Poincaré characteristic $\chi(L^{\otimes t})$ of $L^{\otimes t}$ is a polynomial in t of degree n (see [13, chapter I, §1]), and we put

$$\chi(L^{\otimes t})=\sum_{j=0}^{n}\chi_j(X,L)\binom{t+j-1}{j}.$$

(2) Let Y be a smooth projective variety of dimension $i\geq 1$, let $\mathcal{T}_Y$ be the tangent bundle of Y and let Ω_Y be the dual bundle of $\mathcal{T}_Y$. For every integer j with $0\leq j\leq i$, we put

$$h_{i,j}(c_1(Y),\cdots,c_i(Y)):=\chi(\Omega_Y^j)=\int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y).$$

(Here $\mathrm{ch}(\Omega_Y^j)$ (resp. $\mathrm{Td}(\mathcal{T}_Y)$) denotes the Chern character of Ω_Y^j (resp. the Todd class of $\mathcal{T}_Y$). See [12, example 3.2.3 and example 3.2.4].)

(3) Let X be a smooth projective variety of dimension n. For every integers i and j with $0\leq j\leq i\leq n$, we put

$$H_1(X;i,j):=\begin{cases}\displaystyle\sum_{s=0}^{i-j-1}(-1)^s h^s(\Omega_X^j) & \text{if } j\neq i,\\ 0 & \text{if } j=i,\end{cases}$$

$$H_2(X;i,j):=\begin{cases}\displaystyle\sum_{t=0}^{j-1}(-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j\neq 0,\\ 0 & \text{if } j=0.\end{cases}$$

Definition 2.2.1 ([3], [5] and [6]) Let (X,L) be a polarized manifold of dimension n, and let i and j be integers with $0\leq j\leq i\leq n$. (Here we use Notation 2.2.1.)

(i) The *ith sectional H-arithmetic genus $\chi_i^H(X,L)$ of* (X,L) is defined as follows:

$$\chi_i^H(X,L):=\chi_{n-i}(X,L).$$

(ii) The *ith sectional geometric genus $g_i(X,L)$ of* (X,L) is defined as follows:

$$g_i(X,L):=(-1)^i(\chi_{n-i}(X,L)-\chi(\mathcal{O}_X))+\sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

(iii) The *ith sectional Euler number* $e_i(X,L)$ *of* (X,L) is defined by the following:

$$e_i(X,L) := C_i^{n,1}(X,L)L^{n-i}.$$

(iv) The *ith sectional Betti number* $b_i(X,L)$ *of* (X,L) is defined by the following:

$$b_i(X,L) := \begin{cases} e_0(X,L) & \text{if } i=0, \\ (-1)^i \left(e_i(X,L) - \sum_{j=0}^{i-1} 2(-1)^j h^j(X,\mathbb{C}) \right) & \text{if } 1 \le i \le n. \end{cases}$$

(v) The *ith sectional Hodge number* $h_i^{j,i-j}(X,L)$ *of type* $(j,i-j)$ *of* (X,L) is defined by the following:

$$h_i^{j,i-j}(X,L) := (-1)^{i-j}\left\{ w_i^j(X,L) - H_1(X;i,j) - H_2(X;i,j) \right\},$$

where

$$w_i^j(X,L) := \begin{cases} h_{i,j}(C_1^{n,1}(X,L),\cdots,C_i^{n,1}(X,L))L^{n-i}, & \text{if } i>0, \\ L^n, & \text{if } i=0. \end{cases}$$

Remark 2.2.1 Let (X,L) be a polarized manifold of dimension n and let i be an integer with $1 \le i \le n-1$. Assume that there exists a sequence of smooth projective varieties $X = X_0 \supset X_1 \supset \cdots \supset X_{n-i}$ such that $\dim X_k = n-k$ and $X_k \in |L|_{X_{k-1}}|$ for $1 \le k \le n-i$. Then

$$\chi_i^H(X,L) = \chi(\mathcal{O}_{X_{n-i}}), \quad g_i(X,L) = h^i(\mathcal{O}_{X_{n-i}}) = h^0(\Omega^i_{X_{n-i}})$$

$$e_i(X,L) = e(X_{n-i}),^1 \quad b_i(X,L) = h^i(X_{n-i},\mathbb{C})$$

$$h_i^{j,i-j}(X,L) = h^{i-j}(\Omega^j_{X_{n-i}}) \text{ for every integer } j \text{ with } 0 \le j \le i.$$

3 Sectional invariants of generalized polarized manifolds

In this section, we will define sectional invariants of generalized polarized manifolds, which are thought to be a generalization of sectional invariants of polarized manifolds stated in 2.2. In this section we assume the following unless otherwise mentioned.

Setting 3.1 Let $(X,\mathcal{E})$ be an n-dimensional generalized polarized manifold with rank $\mathcal{E} = r \le n$.

The following fact will be used later.

Fact 3.1 *There exists a very ample line bundle* A *on* X *such that* $\mathcal{E}\otimes A^{\otimes t}$ *is ample and spanned by any positive integer* t*. We set* $\mathcal{E}(t) := \mathcal{E}\otimes A^{\otimes t}$*. Furthermore there exists a general section of* $H^0(\mathcal{E}(t))$ *whose zero locus* $Z(t)$ *is smooth with* $\dim Z(t) = n-r$.

3.1 c_r-sectional H-arithmetic genera and c_r-sectional Euler numbers

In 3.1, we will define the c_r-sectional H-arithmetic genus (resp. the c_r-sectional Euler number) which is a generalization of the sectional H-arithmetic genus (resp. the sectional Euler number) of polarized manifolds.

[1] $e(X_{n-i})$ denotes the Euler number of X_{n-i}

Definition 3.1.1 The *c_r-sectional H-arithmetic genus* $\chi^H_{n,r}(X,\mathcal{E})$ and the *c_r-sectional Euler number* $e_{n,r}(X,\mathcal{E})$ of $(X,\mathcal{E})$ are defined by the following[2]:

$$\begin{aligned}
\chi^H_{n,r}(X,\mathcal{E}) &:= \mathrm{td}_{n-r}\left(C^{n,r}_1(X,\mathcal{E}),\cdots,C^{n,r}_{n-r}(X,\mathcal{E})\right)c_r(\mathcal{E}).\\
e_{n,r}(X,\mathcal{E}) &:= C^{n,r}_{n-r}(X,\mathcal{E})c_r(\mathcal{E}).
\end{aligned}$$

Remark 3.1.1 If $r=n$, then we see that $\chi^H_{n,r}(X,\mathcal{E})=c_n(\mathcal{E})$ and $e_{n,r}(X,\mathcal{E})=c_n(\mathcal{E})$.

The following shows the geometric meaning of these invariants.

Proposition 3.1.1 *Assume that $r\le n-1$ and there exists a smooth projective variety Z such that $\dim Z=n-r$ and Z is the zero locus of an element of $H^0(\mathcal{E})$. Then*

$$\chi^H_{n,r}(X,\mathcal{E})=\chi(\mathcal{O}_Z),\qquad e_{n,r}(X,\mathcal{E})=e(Z).$$

Proof. First we consider $\chi^H_{n,r}(X,\mathcal{E})$. Then by Remark 2.1.2 we have

$$\begin{aligned}
\chi^H_{n,r}(X,\mathcal{E}) &= \mathrm{td}_{n-r}\left(C^{n,r}_1(X,\mathcal{E}),\cdots,C^{n,r}_{n-r}(X,\mathcal{E})\right)c_r(\mathcal{E})\\
&= \mathrm{td}_{n-r}\left(c_1(Z),\cdots,c_{n-r}(Z)\right)\\
&= \chi(\mathcal{O}_Z).
\end{aligned}$$

Next we consider $e_{n,r}(X,\mathcal{E})$. By Remark 2.1.2 we see that

$$e_{n,r}(X,\mathcal{E})=C^{n,r}_{n-r}(X,\mathcal{E})c_r(\mathcal{E})=c_{n-r}(Z)=e(Z).$$

Hence we get the assertion. □

Proposition 3.1.2 *$\chi^H_{n,r}(X,\mathcal{E})$ and $e_{n,r}(X,\mathcal{E})$ are integers.*

Proof. By definition we see that $e_{n,r}(X,\mathcal{E})$ is an integer. Thus we will show that $\chi^H_{n,r}(X,\mathcal{E})$ is an integer. If $n=r$, then by Remark 3.1.1 we get the assertion. So we assume that $r\le n-1$. Here we use Fact 3.1 and notation in Fact 3.1. Then by [12, Example 3.2.2], we see that for every integer k with $1\le k\le n-r$

$$c_k(\mathcal{E}(t))=c_k(\mathcal{E})+\sum_{j=0}^{k-1}\binom{r-j}{k-j}c_j(\mathcal{E})c_1(A)^{k-j}t^{k-j}.$$

Therefore by Definition 3.1.1, we have $\chi^H_{n,r}(X,\mathcal{E}(t))-\chi^H_{n,r}(X,\mathcal{E})\in\mathbb{Q}[t]$. We put

$$f(t):=\chi^H_{n,r}(X,\mathcal{E}(t))-\chi^H_{n,r}(X,\mathcal{E}).$$

Then there exists a positive integer t_1 such that $f(t_1)$ is an integer. By Fact 3.1 and Proposition 3.1.1 we infer that $\chi^H(X,\mathcal{E}(t_1))$ is an integer. Hence $\chi^H_{n,r}(X,\mathcal{E})$ is an integer. □

3.2 c_r-sectional geometric genera and c_r-sectional Betti numbers

Definition 3.2.1 The *c_r-sectional geometric genus* $g_{n,r}(X,\mathcal{E})$ and the *c_r-sectional Betti number* $b_{n,r}(X,\mathcal{E})$ *of* $(X,\mathcal{E})$ are defined by the following:

$$g_{n,r}(X,\mathcal{E}):=(-1)^{n-r}\chi^H_{n,r}(X,\mathcal{E})+(-1)^{n-r+1}\chi(\mathcal{O}_X)+\sum_{k=0}^{r}(-1)^{r-k}h^{n-k}(\mathcal{O}_X).$$

$$b_{n,r}(X,\mathcal{E}):=\begin{cases}(-1)^{n-r}\left(e_{n,r}(X,\mathcal{E})-\sum_{j=0}^{n-r-1}2(-1)^jh^j(X,\mathbb{C})\right), & \text{if } r<n,\\ e_{n,n}(X,\mathcal{E}), & \text{if } r=n.\end{cases}$$

[2]Here td_{n-r} means the Todd polynomial of weight $n-r$ (see [2, Definition 1.4 (1)]).

Remark 3.2.1 (i) If $r = n$, then we see that $g_{n,r}(X, \mathcal{E}) = c_n(\mathcal{E})$ and $b_{n,r}(X, \mathcal{E}) = c_n(\mathcal{E})$.
(ii) The invariant $g_{n,r}(X, \mathcal{E})$ in Definition 3.2.1 is equal to the invariant $g_{n-r}(X, \mathcal{E})$ in [2, Definition 2.1].

By definition and Proposition 3.1.2, we get the following.

Proposition 3.2.1 *$g_{n,r}(X, \mathcal{E})$ and $b_{n,r}(X, \mathcal{E})$ are integers.*

Moreover we see that $g_{n,r}(X, \mathcal{E})$ and $b_{n,r}(X, \mathcal{E})$ have the following property.

Proposition 3.2.2 *Assume that $r \leq n-1$ and there exists a smooth projective variety Z such that $\dim Z = n - r$ and Z is the zero locus of an element of $H^0(\mathcal{E})$. Then*

$$g_{n,r}(X, \mathcal{E}) = h^{n-r}(\mathcal{O}_Z), \qquad b_{n,r}(X, \mathcal{E}) = h^{n-r}(Z, \mathbb{C}).$$

Proof. First we consider the c_r-sectional geometric genus. Then by [14, 1.3 Theorem], [15, Theorem 1.1 (1.1.3) and (1.1.4)] and Proposition 3.1.1 we have

$$\begin{aligned} g_{n,r}(X, \mathcal{E}) &= (-1)^{n-r}\chi^H_{n,r}(X, \mathcal{E}) - (-1)^{n-r}\sum_{j=0}^{n-r-1}(-1)^j h^j(\mathcal{O}_X) \\ &= (-1)^{n-r}\chi(\mathcal{O}_Z) - (-1)^{n-r}\sum_{j=0}^{n-r-1}(-1)^j h^j(\mathcal{O}_Z) \\ &= h^{n-r}(\mathcal{O}_Z). \end{aligned}$$

Next we consider the c_r-sectional Betti number. By Proposition 3.1.1 we get $e_{n,r}(X, \mathcal{E}) = e(Z)$. By [14, 1.3 Theorem], we obtain $h^j(X, \mathbb{C}) = h^j(Z, \mathbb{C})$ for every integer j with $j \leq n-r-1$. Here we note that $h^j(Z, \mathbb{C}) = h^{2(n-r)-j}(Z, \mathbb{C})$ by the Poincaré duality. Hence

$$\begin{aligned} b_{n,r}(X, \mathcal{E}) &= (-1)^{n-r}\left(e_{n,r}(X, \mathcal{E}) - 2\sum_{j=0}^{n-r-1}(-1)^j h^j(X, \mathbb{C})\right) \\ &= (-1)^{n-r}\left(e(Z) - 2\sum_{j=0}^{n-r-1}(-1)^j h^j(Z, \mathbb{C})\right) \\ &= h^{n-r}(Z, \mathbb{C}). \end{aligned}$$

Hence the assertion is obtained. □

Remark 3.2.2 If $r = n-1$, then $g_{n,n-1}(X, \mathcal{E})$ is the curve genus of $(X, \mathcal{E})$ (see e.g. [1] and [16]).

3.3 c_r-sectional Hodge numbers

Definition 3.3.1 The *c_r-sectional Hodge number $h^{j,n-r-j}_{n,r}(X, \mathcal{E})$ of type $(j, n-r-j)$ of $(X, \mathcal{E})$* is defined by the following[3]:

$$h^{j,n-r-j}_{n,r}(X, \mathcal{E}) := (-1)^{n-r-j}\left\{w^j_{n,r}(X, \mathcal{E}) - H_1(X; n-r, j) - H_2(X; n-r, j)\right\}.$$

Here we set

$$w^j_{n,r}(X, \mathcal{E}) := \begin{cases} h_{n-r,j}(C^{n,r}_1(X, \mathcal{E}), \cdots, C^{n,r}_{n-r}(X, \mathcal{E}))c_r(\mathcal{E}), & \text{if } r < n, \\ c_n(\mathcal{E}), & \text{if } r = n. \end{cases}$$

for every integer j with $0 \leq j \leq n-r$.

[3] See Notation 2.2.1 (2).

Remark 3.3.1 If $r = n$, then we see that $h^{0,0}_{n,n}(X,\mathcal{E}) = c_n(\mathcal{E})$.

Proposition 3.3.1 *Assume that $r \leq n-1$ and there exists a smooth projective variety Z such that $\dim Z = n-r$ and Z is the zero locus of an element of $H^0(\mathcal{E})$. Then*

$$h^{j,n-r-j}_{n,r}(X,\mathcal{E}) = h^{j,n-r-j}(Z)$$

for every integer j with $0 \leq j \leq n-r$.

Proof. First we note that $H_1(X; n-r, j) = H_1(Z; n-r, j)$ and $H_2(X; n-r, j) = H_2(Z; n-r, j)$ by [14, 1.3 Theorem], [15, Theorem 1.1 (1.1.3) and (1.1.4)] since $0 \leq j \leq n-r$. We also note that $w^j_{n,r}(X,\mathcal{E}) = h_{n-r,j}(C^{n,r}_1(X,\mathcal{E}), \cdots, C^{n,r}_{n-r}(X,\mathcal{E}))c_r(\mathcal{E}) = h_{n-r,j}(c_1(Z), \cdots, c_{n-r}(Z)) = \chi(\Omega^j_Z)$. Hence by definition we get

$$\begin{aligned} h^{j,n-r-j}_{n,r}(X,\mathcal{E}) &= (-1)^{n-r-j}\left\{w^j_{n,r}(X,\mathcal{E}) - H_1(X; n-r, j) - H_2(X; n-r, j)\right\} \\ &= (-1)^{n-r-j}\left\{\chi(\Omega^j_Z) - H_1(Z; n-r, j) - H_2(Z; n-r, j)\right\} \\ &= h^{j,n-r-j}(Z). \end{aligned}$$

Therefore we get the assertion. □

Proposition 3.3.2 *$h^{j,n-r-j}_{n,r}(X,\mathcal{E})$ is an integer for every integer j with $0 \leq j \leq n-r$.*

Proof. If $r = n$, then by Remark 3.3.1, we get the assertion. So we assume that $r \leq n-1$. Here we use Fact 3.1 and notation in Fact 3.1. Here we note that by [12, Example 3.1.1] we have

$$s_j(\mathcal{E}(t)^\vee) = \sum_{k=0}^{j}(-1)^{j-k}\binom{r-1+j}{r-1+k}s_k(\mathcal{E}^\vee)c_1(A)^{j-k}t^{j-k}$$

(see also Remark 2.1.1 (a)), the following equality holds.

$$\begin{aligned} & w^j_{n,r}(X,\mathcal{E}(t)) \\ &= \sum_{k=1}^{n-r}\left\{\left(\sum_{(l_1,\cdots,l_{n-r},m_1,\cdots,m_{n-r})\in A(k)} q_{l_1,\cdots,l_{n-r},m_1,\cdots,m_{n-r}}c_1(X)^{l_1}\cdots c_{n-r}(X)^{l_{n-r}}s_1(\mathcal{E}^\vee)^{m_1}\cdots s_{n-r}(\mathcal{E}^\vee)^{m_{n-r}}\right)\right. \\ &\qquad \left.\times(tA)^k\right\}c_r(\mathcal{E}) + w^j_{n,r}(X,\mathcal{E}), \end{aligned}$$

where $q_{l_1,\cdots,l_{n-r},m_1,\cdots,m_{n-r}} \in \mathbb{Q}$ and

$$A(k) := \left\{(l_1,\cdots,l_{n-r},m_1,\cdots,m_{n-r}) \in \mathbb{Z}^{\oplus 2n-2r}_{\geq 0} \;\middle|\; \sum_{u=1}^{n-r}ul_u + \sum_{v=1}^{n-r}vm_v = n-r-k\right\}.$$

Then there exists a positive integer s such that

$$sq_{l_1,\cdots,l_{n-r},m_1,\cdots,m_{n-r}} \in \mathbb{Z}$$

for every $(l_1,\cdots,l_{n-r},m_1,\cdots,m_{n-r})$. Therefore $w^j_{n,r}(X,\mathcal{E}(s)) - w^j_{n,r}(X,\mathcal{E})$ is an integer. Since $\mathcal{E}(s)$ is generated by its global sections, by Fact 3.1 and Proposition 3.3.1 we see that $w^j_{n,r}(X,\mathcal{E}(s))$ is also an integer. Therefore $w^j_{n,r}(X,\mathcal{E}) \in \mathbb{Z}$ and we get the assertion by the definition of $h^{j,n-r-j}_{n,r}(X,\mathcal{E})$. □

4 Fundamental properties of these invariants

In this section, we will study fundamental properties of invariants defined above. In particular, we will consider some relations among them. First of all, we can prove the following theorem in general.

Theorem 4.1 *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension n with rank $\mathcal{E}=r$. Assume that $r\leq n-1$. For every integer j with $0\leq j\leq n-r$, we get the following.*
(i) $b_{n,r}(X,\mathcal{E})=\sum_{k=0}^{n-r}h_{n,r}^{k,n-r-k}(X,\mathcal{E})$.
(ii) $h_{n,r}^{j,n-r-j}(X,\mathcal{E})=h_{n,r}^{n-r-j,j}(X,\mathcal{E})$.
(iii) $h_{n,r}^{n-r,0}(X,\mathcal{E})=h_{n,r}^{0,n-r}(X,\mathcal{E})=g_{n,r}(X,\mathcal{E})$.
(iv) *If $n-r$ is odd, then $b_{n,r}(X,\mathcal{E})$ is even.*

Proof. We use Fact 3.1 and notation in Fact 3.1. Then by Propositions 3.1.1, 3.2.2 and 3.3.1 we have $b_{n,r}(X,\mathcal{E}(t))=h^{n-r}(Z(t),\mathbb{C})$, $h_{n,r}^{j,n-r-j}(X,\mathcal{E}(t))=h^{j,n-r-j}(Z(t))$, and $g_{n,r}(X,\mathcal{E}(t))=h^{n-r}(\mathcal{O}_{Z(t)})$.

By the Hodge theory, we get

$$\begin{aligned}h^{j,n-r-j}(Z(t)) &= h^{n-r-j,j}(Z(t))\\ h^{n-r,0}(Z(t)) &= h^{0,n-r}(Z(t))=h^{n-r}(\mathcal{O}_{Z(t)})\\ h^{n-r}(Z(t),\mathbb{C}) &= \sum_{j=0}^{n-r}h^{j,n-r-j}(Z(t)).\end{aligned}$$

Hence for any positive integer t, we see that

$$h_{n,r}^{j,n-r-j}(X,\mathcal{E}(t)) = h_{n,r}^{n-r-j,j}(X,\mathcal{E}(t)) \tag{1}$$
$$h_{n,r}^{n-r,0}(X,\mathcal{E}(t)) = h_{n,r}^{0,n-r}(X,\mathcal{E}(t))=g_{n,r}(X,\mathcal{E}(t)) \tag{2}$$
$$b_{n,r}(X,\mathcal{E}(t)) = \sum_{j=0}^{n-r}h_{n,r}^{j,n-r-j}(X,\mathcal{E}(t)). \tag{3}$$

Since $b_{n,r}(X,\mathcal{E}(t))$, $h_{n,r}^{j,n-r-j}(X,\mathcal{E}(t))$, $h_{n,r}^{n-r-j,j}(X,\mathcal{E}(t))$ and $g_{n,r}(X,\mathcal{E}(t))$ are polynomials in t, we see that (1), (2) and (3) are true for any integer t. In particular, by putting $t=0$, we get the assertion (i), (ii) and (iii). Furthermore by (i) and (ii), we can prove that $b_{n,r}(X,\mathcal{E})$ is even if $n-r$ is odd. Hence we get the assertion. □

Remark 4.1 (1) In Theorem 4.1, we only assume that $\mathcal{E}$ is ample (not necessarily generated by its global sections).

(2) Let Y be a smooth projective variety of dimension $n-r$. Then

(2.1) (i) in Theorem 4.1 corresponds to $h^{n-r}(Y,\mathbb{C})=\sum_{j=0}^{n-r}h^{j,n-r-j}(Y)$.

(2.2) (ii) in Theorem 4.1 corresponds to $h^{j,n-r-j}(Y)=h^{n-r-j,j}(Y)$ for every integer j with $0\leq j\leq n-r$.

(2.3) (iii) in Theorem 4.1 corresponds to $h^{n-r,0}(Y)=h^{0,n-r}(Y)=h^{n-r}(\mathcal{O}_Y)$.

(2.4) (iv) in Theorem 4.1 corresponds to the following fact: If $n-r$ is odd, then $h^{n-r}(Y,\mathbb{C})$ is even.

(3) If $n-r=1$, then by (i) and (iii) in Theorem 4.1

$$b_{n,r}(X,\mathcal{E})=h_{n,r}^{1,0}(X,\mathcal{E})+h_{n,r}^{0,1}(X,\mathcal{E})=2g_{n,r}(X,\mathcal{E}).$$

Next we prove some inequalities under a special assumption.

Proposition 4.1 *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension n with $1 \le r \le n-1$, where $r = \mathrm{rank}\,\mathcal{E}$. Assume that there exist a smooth projective variety Z of dimension $n-r$ such that Z is the zero locus of an element of $H^0(\mathcal{E})$. Then for every integer j with $1 \le j \le n-r$ the following hold.*

(i) $b_{n,r}(X,\mathcal{E}) \ge 2g_{n,r}(X,\mathcal{E})$.

(ii) $b_{n,r}(X,\mathcal{E}) \ge h^{n-r}(X,\mathbb{C})$.

(iii) $h^{j,n-r-j}_{n,r}(X,\mathcal{E}) \ge h^{j,n-r-j}(X)$.

(iv) *If $n-r=2k$, then* $h^{k,k}_{n,r}(X,\mathcal{E}) \ge 1$.

Proof. (i) By Propositions 3.2.2, 3.3.1 and Theorem 4.1 (i), we obtain

$$\begin{aligned} b_{n,r}(X,\mathcal{E}) &= \sum_{k=0}^{n-r} h^{k,n-r-k}_{n,r}(X,\mathcal{E}) = \sum_{k=0}^{n-r} h^{k,n-r-k}(Z) \\ &\ge h^{0,n-r}(Z) + h^{n-r,0}(Z) = 2h^{n-r}(\mathcal{O}_Z) = 2g_{n,r}(X,\mathcal{E}). \end{aligned}$$

(ii) By Proposition 3.2.2 and [14, 1.3 Theorem], we obtain $b_{n,r}(X,\mathcal{E}) = h^{n-r}(Z,\mathbb{C}) \ge h^{n-r}(X,\mathbb{C})$.
(iii) For every integer j with $0 \le j \le i$, by Proposition 3.3.1 and [14, 1.3 Theorem], we get $h^{j,n-r-j}_{n,r}(X,\mathcal{E}) = h^{j,n-r-j}(Z) \ge h^{j,n-r-j}(X)$.
(iv) By Proposition 3.3.1, we have $h^{k,k}_{n,r}(X,\mathcal{E}) = h^{k,k}(Z) \ge 1$ and we get the assertion. $\square$

5 Sectional invariants of multi-polarized manifolds

In this section, we consider the case where an ample vector bundle $\mathcal{E}$ is a direct sum of ample line bundles. First we define the following notion.

Definition 5.1 Let $L_1,\dots,L_m$ (resp. $\mathcal{E}_1,\dots,\mathcal{E}_m$) be ample line bundles (resp. ample vector bundles with rank $\mathcal{E}_i = r_i$) on X. Then $(X,L_1,\dots,L_m)$ (resp. $(X,\mathcal{E}_1,\dots,\mathcal{E}_m)$) is called a *multi-polarized manifold of type m* (resp. *multi-generalized polarized manifold of type m with rank $(r_1,\dots,r_m)$*).

5.1 Definition

Here we will define sectional invariants of multi-polarized manifolds.

Definition 5.1.1 Let $(X,L_1,\dots,L_{n-i})$ be a generalized polarized manifold of dimension n, where i is an integer with $0 \le i \le n-1$. Then we define the *ith sectional H-arithmetic genus* $\chi^H_i(X,L_1,\dots,L_{n-i})$, the *ith sectional Euler number* $e_i(X,L_1,\dots,L_{n-i})$, the *ith sectional geometric genus* $g_i(X,L_1,\dots,L_{n-i})$, the *ith sectional Betti number* $b_i(X,L_1,\dots,L_{n-i})$ and for every integer j with $0 \le j \le i$, the *ith sectional Hodge number* $h^{j,i-j}_i(X,L_1,\dots,L_{n-i})$ *of type* $(j,i-j)$ are defined as follows.

$$\begin{aligned} \chi^H_i(X,L_1,\dots,L_{n-i}) &:= \chi^H_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}), \\ g_i(X,L_1,\dots,L_{n-i}) &:= g_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}), \\ e_i(X,L_1,\dots,L_{n-i}) &:= e_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}), \\ b_i(X,L_1,\dots,L_{n-i}) &:= b_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}), \\ h^{j,i-j}_i(X,L_1,\dots,L_{n-i}) &:= h^{j,i-j}_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}). \end{aligned}$$

First we prove the following lemma.

Lemma 5.1.1 *Let $(X, \mathcal{E}_1, \dots, \mathcal{E}_m)$ be a multi-generalized polarized manifold of type m with rank $(r_1, \dots, r_m)$ and let $r = \sum_{i=1}^m r_i$. Assume that $r \le n-1$ and there exists a sequence of smooth projective varieties $Z_0 \supset Z_1 \supset \cdots \supset Z_m$ such that $\dim Z_j = n - \sum_{k=1}^j r_k$ and Z_j is the zero locus of an element of $H^0(\mathcal{E}_j|_{Z_{j-1}})$ for every integer j with $1 \le j \le m$, where $Z_0 := X$. Then*

$$\begin{aligned}
\chi^H_{n,r}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m) &= \chi^H_{n-r_1, r-r_1}(Z_1, \mathcal{E}_2|_{Z_1} \oplus \cdots \oplus \mathcal{E}_m|_{Z_1}) \\
&\vdots \\
&= \chi^H_{n-r+r_m, r_m}(Z_{m-1}, \mathcal{E}_m|_{Z_{m-1}}) \\
&= \chi(\mathcal{O}_{Z_m}), \\
e_{n,r}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m) &= e_{n-r_1, r-r_1}(Z_1, \mathcal{E}_2|_{Z_1} \oplus \cdots \oplus \mathcal{E}_m|_{Z_1}) \\
&\vdots \\
&= e_{n-r+r_m, r_m}(Z_{m-1}, \mathcal{E}_m|_{Z_{m-1}}) \\
&= e(Z_m), \\
g_{n,r}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m) &= g_{n-r_1, r-r_1}(Z_1, \mathcal{E}_2|_{Z_1} \oplus \cdots \oplus \mathcal{E}_m|_{Z_1}) \\
&\vdots \\
&= g_{n-r+r_m, r_m}(Z_{m-1}, \mathcal{E}_m|_{Z_{m-1}}) \\
&= h^{n-r}(\mathcal{O}_{Z_m}), \\
b_{n,r}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m) &= b_{n-r_1, r-r_1}(Z_1, \mathcal{E}_2|_{Z_1} \oplus \cdots \oplus \mathcal{E}_m|_{Z_1}) \\
&\vdots \\
&= b_{n-r+r_m, r_m}(Z_{m-1}, \mathcal{E}_m|_{Z_{m-1}}) \\
&= h^{n-r}(Z_m, \mathbb{C}), \\
h^{j,n-r-j}_{n,r}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m) &= h^{j,(n-r_1)-(r-r_1)-j}_{n-r_1, r-r_1}(Z_1, \mathcal{E}_2|_{Z_1} \oplus \cdots \oplus \mathcal{E}_m|_{Z_1}) \\
&\vdots \\
&= h^{j,(n-r+r_m)-r_m-j}_{n-r+r_m, r_m}(Z_{m-1}, \mathcal{E}_m|_{Z_{m-1}}) \\
&= h^{j,n-r-j}(Z_m).
\end{aligned}$$

Proof. First we prove the following lemma.

Claim 5.1.1 *Let X be a smooth projective variety of dimension n and let $\mathcal{F}$ and $\mathcal{G}$ be ample vector bundles on X with $\operatorname{rank} \mathcal{F} = r$ and $\operatorname{rank} \mathcal{G} = s$. Assume that there exists a smooth projective variety Z of dimension $n-r$ such that Z is the zero locus of an element of $H^0(\mathcal{F})$. Then for every integer j with $0 \le j \le n-r-s$*

$$C^{n,r+s}_j(X, \mathcal{F} \oplus \mathcal{G}) c_{r+s}(\mathcal{F} \oplus \mathcal{G}) = C^{n-r,s}_j(Z, \mathcal{G}_Z) c_s(\mathcal{G}_Z).$$

Proof. This can be proved by the following equality.

$$\begin{aligned}
C^{n,r+s}_j(X, \mathcal{F} \oplus \mathcal{G}) c_{r+s}(\mathcal{F} \oplus \mathcal{G}) &= \left\{ \sum_{k=0}^{j} c_k(X) s_{j-k}((\mathcal{F} \oplus \mathcal{G})^\vee) \right\} c_r(\mathcal{F}) c_s(\mathcal{G}) \\
&= \left\{ \sum_{k_2=0}^{j} \left(\sum_{k_1=0}^{k_2} c_{k_1}(X) s_{k_2-k_1}(\mathcal{F}^\vee) \right) (s_{j-k_2}(\mathcal{G}^\vee)) \right\} c_r(\mathcal{F}) c_s(\mathcal{G})
\end{aligned}$$

$$= \sum_{k_2=0}^{j} c_{k_2}(Z)(s_{j-k_2}(\mathcal{G}_Z^\vee))c_s(\mathcal{G}_Z) = C_j^{n-r,s}(Z,\mathcal{G}_Z)c_s(\mathcal{G}_Z).$$

$\square$

By Definitons 3.1.1, 3.2.1, 3.3.1, Claim 5.1.1, Propositions 3.1.1, 3.2.2 and 3.3.1, we get the assertion of Lemma 5.1.1. $\square$

By Definition 5.1.1 and Lemma 5.1.1, we can prove the following.

Proposition 5.1.1 *Let i be an integer with $1 \leq i \leq n-1$ and let $(X, L_1, \dots, L_{n-i})$ be an n-dimensional multi-polarized manifold of type $n-i$. Assume that there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_j|_{X_{j-1}}|$ for every integer j with $1 \leq j \leq n-i$. Here we set $X_0 := X$. Then for every integer k with $0 \leq k \leq n-i-1$ we have*

$$\begin{aligned}
\chi_i^H(X_k, L_{k+1}|_{X_k}, \dots, L_{n-i}|_{X_k}) &= \chi_i^H(X_{k+1}, L_{k+2}|_{X_{k+1}}, \dots, L_{n-i}|_{X_{k+1}}),\\
g_i(X_k, L_{k+1}|_{X_k}, \dots, L_{n-i}|_{X_k}) &= g_i(X_{k+1}, L_{k+2}|_{X_{k+1}}, \dots, L_{n-i}|_{X_{k+1}}),\\
e_i(X_k, L_{k+1}|_{X_k}, \dots, L_{n-i}|_{X_k}) &= e_i(X_{k+1}, L_{k+2}|_{X_{k+1}}, \dots, L_{n-i}|_{X_{k+1}}),\\
b_i(X_k, L_{k+1}|_{X_k}, \dots, L_{n-i}|_{X_k}) &= b_i(X_{k+1}, L_{k+2}|_{X_{k+1}}, \dots, L_{n-i}|_{X_{k+1}}),\\
h_i^{j,i-j}(X_k, L_{k+1}|_{X_k}, \dots, L_{n-i}|_{X_k}) &= h_i^{j,i-j}(X_{k+1}, L_{k+2}|_{X_{k+1}}, \dots, L_{n-i}|_{X_{k+1}}).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\chi_i^H(X, L_1, \dots, L_{n-i}) &= \chi(\mathcal{O}_{X_{n-i}}),\\
g_i(X, L_1, \dots, L_{n-i}) &= h^i(\mathcal{O}_{X_{n-i}}),\\
e_i(X, L_1, \dots, L_{n-i}) &= e(X_{n-i}),\\
b_i(X, L_1, \dots, L_{n-i}) &= h^i(X_{n-i}, \mathbb{C}),\\
h_i^{j,i-j}(X, L_1, \dots, L_{n-i}) &= h^{j,i-j}(X_{n-i}).
\end{aligned}$$

5.2 Relation between c_r-sectional invariants and invariants defined before

The following proposition shows that the sectional invariants of polarized manifolds in 2.2 are special cases of invariants defined in Definition 5.1.1.

Proposition 5.2.1 *Let i be an integer with $0 \leq i \leq n-1$ and let $(X, L_1, \dots, L_{n-i})$ be a multi-polarized manifold of type $n-i$. Assume that a line bundle L is ample and $L_k = L$ for every integer k with $1 \leq k \leq n-i$. Then we have*

$$\begin{aligned}
\chi_i^H(X, L_1, \dots, L_{n-i}) &= \chi_i^H(X, L), \quad g_i(X, L_1, \dots, L_{n-i}) = g_i(X, L),\\
e_i(X, L_1, \dots, L_{n-i}) &= e_i(X, L), \quad b_i(X, L_1, \dots, L_{n-i}) = b_i(X, L).\\
h_i^{j,i-j}(X, L_1, \dots, L_{n-i}) &= h_i^{j,i-j}(X, L) \quad \text{for every integer } j \text{ with } 0 \leq j \leq i.
\end{aligned}$$

Here $\chi_i^H(X, L)$, $g_i(X, L)$, $e_i(X, L)$, $b_i(X, L)$, $h_i^{j,i-j}(X, L)$ are sectional invariants defined in Definition 2.2.1.

Proof. We will prove the first equality. Other equalities can be also proved by the same argument as the following. Let H be an ample line bundle on X such that $L(t) := L \otimes H^{\otimes t}$ is ample and spanned for any positive integer t. Then there exists a sequence of smooth projective varieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_k \in |L(t)|_{X_{k-1}}|$ for every integer k with $1 \leq k \leq n-i$. Then by Proposition 5.1.1 and [5, Remark 2.1 (4)] we have

$$\chi_i^H(X, \underbrace{L(t), \ldots, L(t)}_{n-i}) = \chi(\mathcal{O}_{X_{n-i}}) = \chi_i^H(X, L(t)).$$

Since $\chi_i^H(X, L(t), \ldots, L(t))$ and $\chi_i^H(X, L(t))$ are polynomials in t, by the same argument as in the proof of Theorem 4.1, we have

$$\chi_i^H(X, L, \ldots, L) = \chi_i^H(X, L(0), \ldots, L(0)) = \chi_i^H(X, L(0)) = \chi_i^H(X, L).$$

Therefore we get the assertion. □

Remark 5.2.1 Under the assumption that X is smooth, we see that $\chi_i^H(X, L_1, \ldots, L_{n-i})$ (resp. $g_i(X, L_1, \ldots, L_{n-i})$) in Definition 5.1.1 is equal to

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X) \quad (\text{resp. } g_i(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X))$$

in [7, Definition 2.1]. We also note that $\chi_i^H(X, L_1, \ldots, L_{n-i})$ (resp. $g_i(X, L_1, \ldots, L_{n-i})$) is defined for any *smooth* projective variety X, but in [7, Definition 2.1], $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X)$ (resp. $g_i(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X)$) was defined for any projective varieties.

We also note the following.

Proposition 5.2.2 *Let X be a smooth projective variety of dimension n.*

(1) *Let $\mathcal{E}$ be an ample vector bundle of rank e on X with $e \leq n$. Let $\mathcal{E}_1 = \mathcal{E}$ and let $\mathcal{E}_2 = \cdots = \mathcal{E}_{n-e-i+1} = c_1(\mathcal{E})$ if $i \leq n-e-1$. Then $g_{n,n-i}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{n-e-i+1})$ is equal to $g_i(X, \mathcal{E})$ which is the ith c_r-sectional geometric genus of multi-polarized manifold $(X, \mathcal{E})$ defined in* [2, Definition 2.1].

(2) *Let $\mathcal{E}$ be an ample vector bundle of rank e on X with $e \leq n-1$ and let H be an ample line bundle on X. Let $\mathcal{E}_1 = \mathcal{E}$ and let $\mathcal{E}_2 = \cdots = \mathcal{E}_{n-e} = H$ if $e \leq n-2$. Then $g_{n,n-1}(X, \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{n-e})$ is equal to the invariant $g(X, \mathcal{E}, H)$ which was defined by Fusi and Lanteri in* [11].

Proof. We are going to prove the assertion (1). We use Fact 3.1.[4] Then by Lemma 5.1.1 and Definition 5.1.1, we see that for every positive integer t

$$\begin{aligned} g_{n,n-i}(X, \mathcal{E}(t) \oplus \underbrace{c_1(\mathcal{E}(t)) \oplus \cdots \oplus c_1(\mathcal{E}(t))}_{n-e-1}) &= g_{n-e,n-e-i}(Z(t), \underbrace{c_1(\mathcal{E}(t))_{Z(t)} \oplus \cdots \oplus c_1(\mathcal{E}(t))_{Z(t)}}_{n-e-1}) \\ &= g_i(Z(t), \underbrace{c_1(\mathcal{E}(t))_{Z(t)}, \cdots, c_1(\mathcal{E}(t))_{Z(t)}}_{n-e-1}). \end{aligned}$$

By Proposition 5.2.1 and [2, Theorem 2.2], we have

$$g_i(Z(t), \underbrace{c_1(\mathcal{E}(t))_{Z(t)}, \cdots, c_1(\mathcal{E}(t))_{Z(t)}}_{n-e-1}) = g_i(Z(t), c_1(\mathcal{E}(t))_{Z(t)}) = g_i(X, \mathcal{E}(t)).$$

Hence we get

$$g_{n,n-i}(X, \mathcal{E}(t) \oplus c_1(\mathcal{E}(t)) \oplus \cdots \oplus c_1(\mathcal{E}(t))) = g_i(X, \mathcal{E}(t))$$

for every positive integer t. By the same argument as in the proof of Theorem 4.1, we see that

$$g_i(X, \mathcal{E}) = g_{n,n-i}(X, \mathcal{E} \oplus c_1(\mathcal{E}) \oplus \cdots \oplus c_1(\mathcal{E})).$$

So we get the assertion of (1). We can also prove (2) by the same argument as the proof of (1). □

[4] Here let $e = \text{rank } \mathcal{E}$.

5.3 Some properties of the sectional Euler numbers, the sectional Betti numbers and the sectional Hodge numbers of multi-polarized manifolds

In [7] and [9], we studied the sectinal H-arithmetic genus and the sectional geometric genus of multi-polarized manifolds (see also Remark 5.2.1). So here, we will study some properties of the sectional Euler numbers, the sectional Betti numbers and the sectional Hodge numbers of multi-polarized manifolds. First we will show Theorem 5.3.1 which is a generalization of [6, Theorem 4.4]. Before this, we need the following.

Definition 5.3.1 Let k be a positive integer.

(1) Let $(X, L_1, \cdots, L_k)$ and $(Y, A_1, \cdots, A_k)$ be n-dimensional multi-polarized manifolds of type k. Then $(X, L_1, \cdots, L_k)$ is called a *simple blowing up of a multi-polarized manifold* $(Y, A_1, \cdots, A_k)$ *of type* k if there exists a blowing up $\pi : X \to Y$ at a point $y \in Y$ such that $L_j = \pi^*(A_j) - E$ and $E|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ for every integer j with $1 \leq j \leq k$, where $E \cong \mathbb{P}^{n-1}$ is the exceptional effective divisor.

(2) A multi-polarized manifold $(\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$ of type k is called a *reduction of* $(X, L_1, \cdots, L_k)$ if there exists a birational morphism

$$\pi : (X, L_1, \cdots, L_k) \to (\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$$

such that π is a composite of simple blowing ups and $(\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$ is not a simple blowing up of another multi-polarized manifold of type k. This π is called the *reduction map.*

Remark 5.3.1 Let $(X, L_1, \ldots, L_k)$ be a multi-polarized manifold of type k, where k is an integer with $1 \leq k \leq n-1$.
(i) If $(X, L_1, \ldots, L_k)$ is not a simple blowing up of another multi-polarized manifold of type k, then we regard $(X, L_1, \ldots, L_k)$ as a reduction of itself. Then there always exists a reduction of $(X, L_1, \ldots, L_k)$.
(ii) Let $(X, L_1, \ldots, L_k)$ be a simple blowing up of $(Y, H_1, \ldots, H_k)$. Let π be its birational morphism and let E be its exceptional divisor. Assume that there exists a smooth projective variety $X_1 \in |L_1|$. Then $Y_1 := \pi(X_1)$ is also a smooth projective variety of dimension $n-1$ and $Y_1 \in |H_1|$. By the same argument as in [8, Proposition 2.1] we see that $(E_1, -E_1|_{E_1}) \cong (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1))$ and $(L_j)_{X_1} = (\pi_{X_1})^*(H_j|_{Y_1}) - E_1$ for every integer j with $2 \leq j \leq k$, where $E_1 := E \cap X_1$. Hence $(X_1, L_2|_{X_1}, \ldots, L_k|_{X_1})$ is a simple blowing up of $(Y_1, H_2|_{Y_1}, \ldots, H_k|_{Y_1})$ and $\pi|_{X_1} : X_1 \to Y_1$ is its birational morphism. By repeating this process, we see that if there exists a smooth projective variety $X_{l+1} \in |L_{l+1}|_{X_l}|$ of dimension $n-l-1$ for every integer l with $1 \leq l \leq k-2$, then $Y_{l+1} := (\pi|_{X_l})(X_{l+1})$ is also a smooth projective variety of dimension $n-l-1$ and $Y_{l+1} \in |H_{l+1}|_{Y_l}|$, and we infer that $(X_{l+1}, L_{l+2}|_{X_{l+1}}, \ldots, L_k|_{X_{l+1}})$ is a simple blowing up of $(Y_{l+1}, H_{l+2}|_{Y_{l+1}}, \ldots, H_k|_{Y_{l+1}})$ and $\pi|_{X_{l+1}} : X_{l+1} \to Y_{l+1}$ is its birational morphism.

Proposition 5.3.1 *Let $(X, L_1, \cdots, L_{n-i})$ be a multi-polarized manifold of type $n-i$ with $\dim X = n$, let $(Y, H_1, \cdots, H_{n-i})$ be a reduction of $(X, L_1, \cdots, L_{n-i})$ and let $\pi : X \to Y$ be its reduction map. Let γ be the number of points blown up under the reduction map. Let i and j be integers with $0 \leq j \leq i$ and $0 \leq i \leq n-1$. Then*

(a)

$$e_i(X, L_1, \cdots, L_{n-i}) = e_i(Y, H_1, \cdots, H_{n-i}) + (i-1)\gamma.$$

(b)

$$b_i(X, L_1, \cdots, L_{n-i}) = \begin{cases} b_i(Y, H_1, \cdots, H_{n-i}) & \text{if } i \text{ is odd,} \\ b_i(Y, H_1, \cdots, H_{n-i}) + \gamma & \text{if } i \text{ is even with } i \geq 2, \\ b_0(Y, H_1, \cdots, H_{n-i}) - \gamma & \text{if } i = 0. \end{cases}$$

(c)

$$h_i^{j,i-j}(X,L_1,\cdots,L_{n-i}) = \begin{cases} h_i^{j,i-j}(Y,H_1,\cdots,H_{n-i}) & \textit{if } 1 \le i \textit{ and } 2j \ne i, \\ h_i^{j,i-j}(Y,H_1,\cdots,H_{n-i}) + \gamma & \textit{if } 1 \le i \textit{ and } 2j = i, \\ h_0^{0,0}(Y,H_1,\cdots,H_{n-i}) - \gamma & \textit{if } i = 0. \end{cases}$$

Proof. First we note that it suffices to consider the case where $(X, L_1, \cdots, L_{n-i})$ is a simple blowing up of $(Y, H_1, \cdots, H_{n-i})$. Let $\pi : X \to Y$ be its morphism. Then $L_j = \pi^*(H_j) - E$ holds for every integer j with $1 \le j \le n-i$, where E is the exceptional divisor. Let $H_j(t) := H_j^{\otimes t}$ and $L_j(t) := \pi^*(H_j(t)) - E$. By the same argument as in the proof of [8, Claim 2.1], there exists a positive integer p such that $H_j(t)$ and $L_j(t)$ are ample and spanned for every integers j and t with $1 \le j \le n-i$ and $t \ge p$. By Remark 5.3.1 (ii), for every integer k with $1 \le k \le n-i-2$ there exists a smooth projective variety $X_{k+1}(t) \in |L_{k+1}(t)|_{X_k(t)}|$ of dimension $n-k-1$ such that $Y_{k+1}(t) := (\pi|_{X_k(t)})(X_{k+1}(t))$ is also a smooth projective variety of dimension $n-k-1$ and $Y_{k+1}(t) \in |H_{k+1}(t)|_{Y_k(t)}|$, and we see that $(X_{k+1}(t), L_{k+2}(t)|_{X_{k+1}(t)}, \ldots, L_{n-i}(t)|_{X_{k+1}(t)})$ is a simple blowing up of $(Y_{k+1}(t), H_{k+2}(t)|_{Y_{k+1}(t)}, \ldots, H_{n-i}(t)|_{Y_{k+1}(t)})$ and $\pi|_{X_{k+1}(t)} : X_{k+1}(t) \to Y_{k+1}(t)$ is its birational morphism. Therefore by Proposition 5.1.1, [6, Theorem 3.2] and [8, Proposition 2.2], we see that the following hold for every integer t with $t \ge p$.

$$\begin{aligned} e_i(X, L_1(t), \cdots, L_{n-i}(t)) &= e_i(X_{n-i-1}(t), L_{n-i}(t)|_{X_{n-i-1}(t)}) \\ &= e_i(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) + (i-1) \\ &= e_i(Y, H_1(t), \cdots, H_{n-i}(t)) + (i-1), \end{aligned}$$

$$\begin{aligned} b_i(X, L_1(t), \cdots, L_{n-i}(t)) &= b_i(X_{n-i-1}(t), L_{n-i}(t)|_{X_{n-i-1}(t)}) \\ &= \begin{cases} b_i(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) & \text{if } i \text{ is odd,} \\ b_i(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) + 1 & \text{if } i \text{ is even with } i \ge 2, \\ b_0(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) - 1 & \text{if } i = 0, \end{cases} \\ &= \begin{cases} b_i(Y, H_1(t), \cdots, H_{n-i}(t)) & \text{if } i \text{ is odd,} \\ b_i(Y, H_1(t), \cdots, H_{n-i}(t)) + 1 & \text{if } i \text{ is even with } i \ge 2, \\ b_0(Y, H_1(t), \cdots, H_{n-i}(t)) - 1 & \text{if } i = 0, \end{cases} \end{aligned}$$

$$\begin{aligned} h_i^{j,i-j}(X, L_1(t), \cdots, L_{n-i}(t)) &= h_i^{j,i-j}(X_{n-i-1}(t), L_{n-i}(t)|_{X_{n-i-1}(t)}) \\ &= \begin{cases} h_i^{j,i-j}(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) & \text{if } 1 \le i \text{ and } 2j \ne i, \\ h_i^{j,i-j}(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) + 1 & \text{if } 1 \le i \text{ and } 2j = i, \\ h_0^{0,0}(Y_{n-i-1}(t), H_{n-i}(t)|_{Y_{n-i-1}(t)}) - 1 & \text{if } i = 0, \end{cases} \\ &= \begin{cases} h_i^{j,i-j}(Y, H_1(t), \cdots, H_{n-i}(t)) & \text{if } 1 \le i \text{ and } 2j \ne i, \\ h_i^{j,i-j}(Y, H_1(t), \cdots, H_{n-i}(t)) + 1 & \text{if } 1 \le i \text{ and } 2j = i, \\ h_0^{0,0}(Y, H_1(t), \cdots, H_{n-i}(t)) - 1 & \text{if } i = 0. \end{cases} \end{aligned}$$

Here we note that $e_i(X, L_1(t), \cdots, L_{n-i}(t))$, $e_i(Y, H_1(t), \cdots, H_{n-i}(t))$, $b_i(X, L_1(t), \cdots, L_{n-i}(t))$, $b_i(Y, H_1(t), \cdots, H_{n-i}(t))$, $h_i^{j,i-j}(X, L_1(t), \cdots, L_{n-i}(t))$ and $h_i^{j,i-j}(Y, H_1(t), \cdots, H_{n-i}(t))$ are polynomials in t. Hence we see that the above equalities hold for the case of $t = 1$, and we get the assertion. □

Next we consider a lower bound for the second sectional Euler numbers of multi-polarized manifolds. First of all, we will give the formula for the sectional Euler number of multi-polarized manifolds.

Remark 5.3.2 We note that

$$s_k((L_1 \oplus \cdots \oplus L_r)^\vee) = (-1)^k \sum_{(p_1,\dots,p_r)\in H(k)} (L_1^{p_1} \cdots L_r^{p_r}), \tag{4}$$

where we set

$$H(k) = \left\{ (p_1, \dots, p_r) \in \mathbb{Z}_{\geq 0}^{\oplus r} \;\middle|\; \sum_{j=1}^{r} p_j = k \right\}.$$

Hence, by Definition 5.1.1 and (4) we see that

$$e_i(X, L_1, \dots, L_{n-i}) = \sum_{k=0}^{i} (-1)^{i-k} c_k(X) \left(\sum_{(p_1,\dots,p_{n-i})\in H(i-k)} L_1^{p_1} \cdots L_{n-i}^{p_{n-i}} \right) L_1 \cdots L_{n-i}. \tag{5}$$

Theorem 5.3.1 *Let $(X, L_1, \dots, L_{n-2})$ be a multi-polarized manifold of type $n-2$ with $\dim X = n \geq 3$. Assume that $\kappa(X) \geq 0$. Let $(M, A_1, \dots, A_{n-2})$ be a reduction of $(X, L_1, \dots, L_{n-2})$ and let γ be the number of points blown up under the reduction map. Then the following hold.*

(i)

$$\begin{aligned}
&e_2(X, L_1, \dots, L_{n-2}) \\
&\geq \frac{1}{2n} \left(\sum_{j=1}^{n-2} L_j \right)^2 L_1 \cdots L_{n-2} + \frac{1}{2} \left(\sum_{j=1}^{n-2} L_j^2 \right) L_1 \cdots L_{n-2} + \left(\frac{(n-1)(n-2)}{n} + 1 \right) \gamma \\
&\geq \frac{(n-1)(n-2)}{n} (\gamma + 1) + \gamma.
\end{aligned}$$

(ii)

$$b_2(X, L_1, \dots, L_{n-2}) \geq 4q(X) + \gamma \geq \gamma.$$

Proof. Since $\kappa(X) \geq 0$, we see that $K_M + A_1 + \cdots + A_{n-2}$ is nef and $(n-2)$-big by [9, Theorem 5.2.1]. Hence by [4, Theorem 2.1] we have

$$c_2(M) A_1 \cdots A_{n-2} \geq -\frac{n-1}{n} K_M \left(\sum_{j=1}^{n-2} A_j \right) A_1 \cdots A_{n-2} - \frac{n-1}{2n} \left(\sum_{j=1}^{n-2} A_j \right)^2 A_1 \cdots A_{n-2}. \tag{6}$$

(i) By Proposition 5.3.1, the equality (5) in Remark 5.3.2 and (6) above we have

$$\begin{aligned}
&e_2(X, L_1, \dots, L_{n-2}) \\
&= e_2(M, A_1, \dots, A_{n-2}) + \gamma \\
&= \sum_{k=0}^{2} (-1)^{2-k} c_k(M) \left(\sum_{(p_1,\dots,p_{n-2})\in H(2-k)} A_1^{p_1} \cdots A_{n-2}^{p_{n-2}} \right) A_1 \cdots A_{n-2} + \gamma \\
&\geq \frac{1}{n} K_M \left(\sum_{j=1}^{n-2} A_j \right) A_1 \cdots A_{n-2} + \frac{1}{2n} \left(\sum_{j=1}^{n-2} A_j \right)^2 A_1 \cdots A_{n-2} + \frac{1}{2} \left(\sum_{j=1}^{n-2} A_j^2 \right) A_1 \cdots A_{n-2} + \gamma \\
&\geq \frac{1}{2n} \left(\sum_{j=1}^{n-2} L_j \right)^2 L_1 \cdots L_{n-2} + \frac{(n-2)^2}{2n}\gamma + \frac{1}{2} \left(\sum_{j=1}^{n-2} L_j^2 \right) L_1 \cdots L_{n-2} + \frac{n-2}{2}\gamma + \gamma.
\end{aligned}$$

Since

$$\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} \geq 2\binom{n-2}{2} + n - 2 \quad \text{and} \quad \left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \geq n-2,$$

we have

$$\frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \geq \frac{(n-1)(n-2)}{n}.$$

Hence we get the assertion of (i).
(ii) Next we will prove the inequality (ii). If $n = 3$, then this is true by the proof of [6, Theorem 4.4]. So we assume that $n \geq 4$. First we note that by Definition 5.1.1, Proposition 5.3.1, the equality (5) in Remark 5.3.2 and (6) above we have

$$\begin{aligned} b_2(X, L_1, \ldots, L_{n-2}) &= c_2(M)A_1 \cdots A_{n-2} + K_M\left(\sum_{j=1}^{n-2} A_j\right) A_1 \cdots A_{n-2} \\ &\quad + \sum_{i \leq j} (A_i A_j) A_1 \cdots A_{n-2} + 4q(M) - 2 + \gamma. \end{aligned} \tag{7}$$

(ii.a) Assume that $K_M A_j A_1 \cdots A_{n-2} = 0$ for some j. Then $K_M \equiv 0$ because $\kappa(X) \geq 0$ and each A_j is ample. Hence $c_2(M)A_1 \cdots A_{n-2} \geq 0$ by a Miyaoka's result ([17, Theorem 6.6]). So by (7) we get

$$b_2(X, L_1, \ldots, L_{n-2}) \geq \sum_{i \leq j} (A_i A_j) A_1 \cdots A_{n-2} + 4q(M) - 2 + \gamma.$$

Since we assume that $n \geq 4$, we have $\sum_{i \leq j}(A_i A_j)A_1 \cdots A_{n-2} \geq 3$. Hence $b_2(M, A_1, \ldots, A_{n-2}) \geq 4q(M) + 1 + \gamma > 4q(M) + \gamma$.
(ii.b) Assume that $K_M A_j A_1 \cdots A_{n-2} \geq 1$ for every j. Then by (6) and (7) we have

$$\begin{aligned} b_2(X, L_1, \ldots, L_{n-2}) &\geq \frac{1}{n} K_M \left(\sum_{j=1}^{n-2} A_j\right) A_1 \cdots A_{n-2} + \frac{n+1}{2n}\left(\sum_{j=1}^{n-2} A_j^2\right) A_1 \cdots A_{n-2} \\ &\quad + \frac{1}{n}\sum_{i<j} (A_i A_j) A_1 \cdots A_{n-2} + 4q(M) - 2 + \gamma \\ &\geq 4q(M) + (n-4) + \gamma \geq 4q(M) + \gamma \end{aligned}$$

because $n \geq 4$, $(\sum_{j=1}^{n-2} A_j^2)A_1 \cdots A_{n-2} \geq n-2$ and $\sum_{i<j}(A_i A_j)A_1 \cdots A_{n-2} \geq \binom{n-2}{2}$. So we get the assertion (ii) because $q(M) = q(X)$. □

6 Problems and Conjectures

In this section, we will provide some conjectures and problems. First we propose the following.

Conjecture 6.1 *Let X be a smooth projective variety of dimension n, $\mathcal{E}$ an ample vector bundle of rank r on X. Assume that $r \leq n-1$. Then the following inequality hold.*

$$g_{n,r}(X, \mathcal{E}) \geq 0, \quad b_{n,r}(X, \mathcal{E}) \geq 0.$$

The following problem is very interesting in view of classification theory.

Problem 6.1 *Classify n-dimensional generalized polarized manifold $(X, \mathcal{E})$ with rank $\mathcal{E} \leq n$ by the value of c_r-sectional invariants defined in Section* 3.

More strongly, we can propose the following conjecture by considering Proposition 4.1.

Conjecture 6.2 *Let X be a smooth projective variety of dimension n, $\mathcal{E}$ an ample vector bundle of rank r on X. Assume that $r \leq n-1$. Then for every integer j with $0 \leq j \leq n-r$ the following hold.*

(i) $b_{n,r}(X, \mathcal{E}) \geq 2g_{n,r}(X, \mathcal{E})$. (ii) $b_{n,r}(X, \mathcal{E}) \geq h^{n-r}(X, \mathbb{C})$.

(iii) $h^{j,n-r-j}_{n,r}(X, \mathcal{E}) \geq h^{j,n-r-j}(X)$. (iv) *If $n-r=2k$, then* $h^{k,k}_{n,r}(X, \mathcal{E}) \geq 1$.

If $\mathcal{E}$ is an ample vector bundle of rank r with $0 < i = n-r$, then c_r-sectional invariants of $(X, \mathcal{E})$ are thought to reflect some properties of i-dimensional manifolds from Propositions 3.1.1, 3.2.2 and 3.3.1. In particular we can propose the following problems for the case $i = 2$.

Problem 6.2 *Let X be a smooth projective variety of dimension n, $\mathcal{E}$ an ample vector bundle of rank r on X. Assume that $n-r=2$. Then generalize the theory of surfaces in view of c_r-sectional invariants of $(X, \mathcal{E})$.*

For example, the following is an answer for this problem. We can regard the following theorem as an analogue of Noether's equality.

Theorem 6.1 *Let $(X, \mathcal{E})$ be a generalized polarized manifold of dimension n with rank $\mathcal{E} = n-2$. Then*

$$12\chi^H_{n,n-2}(X, \mathcal{E}) = (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) + e_{n,n-2}(X, \mathcal{E}).$$

Proof. We use Fact 3.1 and notation in Fact 3.1. Let

$$F(t) := (K_X + c_1(\mathcal{E}(t)))^2 c_{n-2}(\mathcal{E}(t)) + e_{n,n-2}(X, \mathcal{E}(t)), \quad G(t) := 12\chi^H_{n,n-2}(X, \mathcal{E}(t)).$$

Then $F(t)$ and $G(t)$ are polynomials in t. For every positive integer p, by Remark 2.1.2 and Proposition 3.1.1 we have $F(p) = (K_X + c_1(\mathcal{E}(p)))^2 c_{n-2}(\mathcal{E}(p)) + e_{n,n-2}(X, \mathcal{E}(p)) = (K_{Z(p)})^2 + e(Z(p))$. Since $Z(p)$ is a smooth projective surface, by Noether's equality we have $(K_{Z(p)})^2 + e(Z(p)) = 12\chi(\mathcal{O}_{Z(p)})$. So by Proposition 3.1.1, for every positive integer p, we have $F(p) = (K_{Z(p)})^2 + e(Z(p)) = 12\chi(\mathcal{O}_{Z(p)}) = 12\chi^H_{n,n-2}(X, \mathcal{E}(p)) = G(p)$. Hence we see that this equality also holds for $t=0$ because $F(t)$ and $G(t)$ are polynomials in t. Therefore we get the assertion. $\square$

Moreover, we can propose the following conjectures specifically.

Conjecture 6.3 *Let X be a smooth projective variety of dimension n, $\mathcal{E}$ an ample vector bundle of rank $n-2$ on X.*

(i) **(Analogue of Castelnuovo's theorem)** *If $\kappa(K_X + c_1(\mathcal{E})) \geq 0$ (resp. ≥ 2), then*

$$\chi^H_{n,n-2}(X, \mathcal{E}) \geq 0 \quad (\textit{resp.} > 0).$$

(ii) **(Analogue of Bogomolov-Miyaoka-Yau's theorem)** *If $\kappa(K_X + c_1(\mathcal{E})) \geq 2$, then*

$$9\chi^H_{n,n-2}(X, \mathcal{E}) \geq (K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}).$$

(iii) **(Analogue of Noether's inequality)** *If $K_X + c_1(\mathcal{E})$ is nef and $\kappa(K_X + c_1(\mathcal{E})) \geq 2$, then*

$$(K_X + c_1(\mathcal{E}))^2 c_{n-2}(\mathcal{E}) \geq 2g_{n,n-2}(X, \mathcal{E}) - 4.$$

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