On generalized polarized manifolds of which the second c_r-sectional geometric genus is equal to $h^2(\mathcal{O})+1$. *†‡

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Abstract

Let $(X,\mathcal{E})$ be a generalized polarized manifold of $\dim X = n \geq 3$ and $\mathrm{rank}(\mathcal{E}) = r \geq 2$. Assume that $\mathcal{E}$ is very ample and $n-r \geq 3$. In this paper we classfy $(X,\mathcal{E})$ with $g_2(X,\mathcal{E}) = h^2(\mathcal{O}_X)+1$, where $g_2(X,\mathcal{E})$ is the second c_r-sectional geometric genus of $(X,\mathcal{E})$.

1 Introduction.

Let X be a projective variety of $\dim X = n$, and let L be an ample (resp. a nef and big) line bundle on X. Then we call the pair (X,L) a *polarized* (resp. *quasi-polarized*) *variety*, and (X,L) is called a polarized (resp. quasi-polarized) *manifold* if X is smooth. In [6], we gave a new invariant of (X,L) which is called the *i-th sectional geometric genus* $g_i(X,L)$ *of* (X,L) for every integer i with $0 \leq i \leq n$. We note that $g_i(X,L)$ is a generalization of the degree L^n and the sectional genus $g(L)$. (Namely $g_0(X,L) = L^n$ and $g_1(X,L) = g(L)$.) Here we recall the reason why we call this invariant the sectional geometric genus. Let (X,L) be a quasi-polarized manifold of dimension $n \geq 2$ with $\mathrm{Bs}|L| = \emptyset$, where $\mathrm{Bs}|L|$ is the base locus of $|L|$. Let i be an integer with $1 \leq i \leq n$, and let Y be the transversal intersection of general $n-i$ elements of $|L|$. In this case Y is a smooth projective variety of dimension i. Then we can prove that $g_i(X,L) = h^i(\mathcal{O}_Y)$, that is, $g_i(X,L)$ is the geometric genus of Y.

**Key words and phrases.* Ample vector bundles, generalized polarized manifold, sectional geometric genus, c_r-sectional geometric genus.

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In [6], we study some fundamental properties of the i-th sectional geometric genus. We were able to generalize some problems about the sectional genus to the case of the sectional geometric genus. For example, in [6], we proposed the following conjecture:

Conjecture 1.1 *Let (X, L) be a quasi-polarized manifold of* $\dim X = n$*. For every integer i with $0 \leq i \leq n$, $g_i(X, L) \geq h^i(\mathcal{O}_X)$ holds.*

Here we note that if $i = 0$, then this is true because $g_0(X, L) = L^n \geq 1 = h^0(\mathcal{O}_X)$. If $i = 1$, then this is a Fujita's conjecture. (See [5], Chapter II, (13.7) or [2], Question 7.2.11.) Hence we can regard the inequality $g(L) \geq h^1(\mathcal{O}_X)$ as a generalization of the inequality $L^n \geq 1$. In [6], we proved that this conjecture is true if $\mathrm{Bs}|L| = \emptyset$. Moreover we classified polarized manifolds (X, L) which satisfy the following properties:

(A) $\dim X \geq 3$, $\mathrm{Bs}|L| = \emptyset$, and $g_2(X, L) = h^2(\mathcal{O}_X)$ (see [6], Corollary 3.5 or see Theorem 1.1 below),

(B) $\dim X \geq 3$, L is very ample, and $g_2(X, L) = h^2(\mathcal{O}_X) + 1$ (see [6], Theorem 3.6).

In a future paper, we will classify polarized manifolds (X, L) such that L is very ample and $g_2(X, L) - h^2(\mathcal{O}_X) \leq 5$. In [7] we study the conjecture for the case where $0 \leq \dim \mathrm{Bs}|L| \leq n - 1$.

Furthermore in [6] we proved the following which is analogous to a theorem of Sommese ([14], Theorem 4.1):

Theorem 1.1 (See [6], Corollary 3.5.) *Let (X, L) be an n-dimensional polarized manifold. Assume that $n \geq 3$ and L is spanned. Then the following are equivalent:*

(A) $g_2(X, L) = h^2(\mathcal{O}_X)$.

(B) $h^0(K_X + (n - 2)L) = 0$.

(C) $\kappa(K_X + (n - 2)L) = -\infty$.

(D) *$K_{X'} + (n - 2)L'$ is not nef, where (X', L') is a reduction of (X, L).*

(E) *(X, L) is one of the following types:*

  (1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
  (2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
  (3) *A scroll over a smooth curve.*
  (4) *$K_X \sim -(n - 1)L$, that is, (X, L) is a Del Pezzo manifold.*
  (5) *A hyperquadric fibration over a smooth curve.*

(6) *A scroll over a smooth surface.*

(7) *Let (X', L') be a reduction of (X, L).*

(7-1) $n = 4$, $(X', L') = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

(7-2) $n = 3$, $(X', L') = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

(7-3) $n = 3$, $(X', L') = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

(7-4) $n = 3$, *X' is a $\mathbb{P}^2$-bundle over a smooth curve C with* $(F', L'|_{F'}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber F' of it.*

In this way, it is interesting and very important to study the sectional geometric genus, and we hope that by using this invariant we can study polarized manifolds more deeply.

In [8], we considered the case of ample vector bundles. Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle of $\mathrm{rank}(\mathcal{E}) = r$. Then the pair $(X, \mathcal{E})$ is called a *generalized polarized manifold*. Here we assume that $1 \leq r \leq n-1$. In [8], for every integer i with $0 \leq i \leq n-r$, we gave a vector bundle's version of the i-th sectional geometric genus, which is called the *i-th c_r-sectional geometric genus of generalized polarized manifolds* $(X, \mathcal{E})$ (see Definition 2.3). Here we note that if $r = 1$, then this is the i-th sectional geometric genus of polarized manifolds. Moreover this is a generalization of the c_r-sectional genus which was defined by Ishihara ([10]). Namely $g_1(X, \mathcal{E})$ is the c_r-sectional genus. (See Theorem 2.1.) Here we note that the c_r-sectional genus is a generalization of the curve genus which was defined by Ballico [1]. Therefore the i-th c_r-sectional geometric genus is a generalization of several important invariants.

Furthermore assume that $\mathcal{E}$ is an ample vector bundle of $\mathrm{rank}(\mathcal{E}) = r \geq 2$ on X with $n - r \geq 1$ such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is a submanifold of X of the expected dimension $n-r$. Then $g_i(X, \mathcal{E}) = g_i(Z, c_1(\mathcal{E}|_Z))$ (see Theorem 2.2). (Here we note that if $\mathcal{E}$ is an ample and spanned vector bundle of $\mathrm{rank}(\mathcal{E}) = r$ with $n - r \geq 1$, then the above assumption is satisfied.)

Let $(X, \mathcal{E})$ be a generalized polarized manifold of $\dim X = n$ and $\mathrm{rank}(\mathcal{E}) = r$ with $n - r \geq 1$ such that $\mathcal{E}$ is ample and spanned. Then in [8] we proved that $g_i(X, \mathcal{E}) \geq h^i(\mathcal{O}_X)$ for every integer i with $0 \leq i \leq n-r$. Moreover if in [8], Theorem 2.7, we classified $(X, \mathcal{E})$ with $g_2(\mathcal{E}) = h^2(\mathcal{O}_X)$, $r \geq 2$, and $n - r \geq 3$.

In this paper, for a very ample vector bundle $\mathcal{E}$ on X of $\mathrm{rank}(\mathcal{E}) = r$ with $n - r \geq 3$, we will classify $(X, \mathcal{E})$ with $g_2(X, \mathcal{E}) = h^2(\mathcal{O}_X) + 1$. Main result is Theorem 3.1.

2 Preliminaries.

Proposition 2.1 *Let $x_0 = 1$ and let x_i be an indeterminate of weight i for every integer i with $i \geq 1$. For any non-negative integer k, there exist unique polynomials of weight k, $T_k \in \mathbb{Q}[x_1, \cdots, x_k]$, such that the following properties hold:*

(1) $T_0 = 1$.

(2) *For any formal power series* $\sum_{i=0}^{\infty} x_i t^i$, *we put*

$$\mathrm{td}_t(\sum_{i=0}^{\infty} x_i t^i) = \sum_{k=0}^{\infty} T_k(x_1, \cdots, x_k)t^k,$$

where t *is an indeterminate.*
If

$$\sum_{i=0}^{\infty} x_i t^i = \left(\sum_{i=0}^{\infty} y_i t^i\right)\left(\sum_{i=0}^{\infty} z_i t^i\right),$$

then

$$\mathrm{td}_t(\sum_{i=0}^{\infty} x_i t^i) = \left(\mathrm{td}_t(\sum_{i=0}^{\infty} y_i t^i)\right)\left(\mathrm{td}_t(\sum_{i=0}^{\infty} z_i t^i)\right).$$

(3) *For the linear expression* $1 + xt$,

$$\mathrm{td}_t(1 + xt) = \frac{xt}{1 - \exp(-xt)}.$$

Proof. See [9], Chapter I, §1. □

Definition 2.1 (1) Polynomials $T_k \in \mathbb{Q}[x_1, \cdots, x_k]$ in Proposition 2.1 is called the *Todd polynomial of weight* k.
(2) Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Let $c_t(\mathcal{F}) = \sum_{i\geq 0} c_i(\mathcal{F})t^i$ be the Chern polynomial of $\mathcal{F}$. We put

$$\mathrm{td}_t(\mathcal{F}) = \mathrm{td}(\sum_{i\geq 0} c_i(\mathcal{F})t^i) = \sum_{k=0}^{\infty} T_k(c_1(\mathcal{F}), \cdots, c_k(\mathcal{F}))t^k,$$

where t is an indeterminate. Furthermore, we put

$$\mathrm{td}_k(c_1(\mathcal{F}), \cdots, c_k(\mathcal{F})) := T_k(c_1(\mathcal{F}), \cdots, c_k(\mathcal{F})),$$

and

$$\mathrm{td}(\mathcal{F}) := \sum_{k=0}^{\infty} \mathrm{td}_k(c_1(\mathcal{F}), \cdots, c_k(\mathcal{F})).$$

Then $\mathrm{td}(\mathcal{F})$ is called the *Todd class of* $\mathcal{F}$.

Definition 2.2 (1) Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Then for every integer j with $j \geq 0$, the j*-th Segre class* $s_j(\mathcal{F})$ *of* $\mathcal{F}$ is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\geq 0} s_j(\mathcal{F})t^j$.
(2) Let X be a smooth projective variety and let $\mathcal{T}_X$ be the tangent bundle of X. Then we put $c_i(X) := c_i(\mathcal{T}_X)$, where $c_i(\mathcal{T}_X)$ is the i-th Chern class of $\mathcal{T}_X$.

Definition 2.3 (See [8], Definition 2.1.) Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X with $1 \leq r \leq n$. Then for every integer i with $0 \leq i \leq n-r$, the *i-th c_r-sectional geometric genus of* $(X, \mathcal{E})$ is defined by the following:

$$\begin{aligned} g_i(X,\mathcal{E}) \ := \ & \sum_{j=0}^{n-r-i} (-1)^{n-r-j} \binom{n-r-i}{j} \\ & \times \sum_{k=0}^{n-r} \left\{ \frac{(-(n-r-i-j)c_1(\mathcal{E}))^{n-r-k}}{(n-r-k)!} \right. \\ & \left. \times \sum_{l=0}^{k} \mathrm{td}_l(c_1(X),\dots,c_l(X))\mathrm{td}_{k-l}(s_1(\mathcal{E}^\vee),\dots,s_{k-l}(\mathcal{E}^\vee)) \right\} c_r(\mathcal{E}) \\ & +(-1)^{i+1}\chi(\mathcal{O}_X) + \sum_{k=0}^{n-i}(-1)^{n-i-k}h^{n-k}(\mathcal{O}_X). \end{aligned}$$

Theorem 2.1 *Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle of $\mathrm{rank}(\mathcal{E}) = r$ on X.*
(1) *If $1 \leq r \leq n-1$, then*

$$g_1(X,\mathcal{E}) = 1 + \frac{1}{2}(K_X + (n-r)c_1(\mathcal{E}))c_1(\mathcal{E})^{n-r-1}c_r(\mathcal{E}).$$

(2) *If $1 \leq r \leq n-2$, then*

$$\begin{aligned} & g_2(X,\mathcal{E}) \\ & = -1 + h^1(\mathcal{O}_X) \\ & \quad +\frac{1}{12}(K_X + (n-r)c_1(\mathcal{E}))(K_X + (n-r-1)c_1(\mathcal{E}))c_r(\mathcal{E})c_1(\mathcal{E})^{n-r-2} \\ & \quad +\frac{1}{12}(c_2(X) + (K_X + c_1(\mathcal{E}))c_1(\mathcal{E}) - c_2(\mathcal{E}))c_r(\mathcal{E})c_1(\mathcal{E})^{n-r-2} \\ & \quad +\frac{n-r-3}{24}(2K_X + (n-r)c_1(\mathcal{E}))c_r(\mathcal{E})c_1(\mathcal{E})^{n-r-1}. \end{aligned}$$

Proof. See [8], Theorem 2.5. □

Definition 2.4 Let X be a smooth projective variety and let $\mathcal{E}$ be a vector bundle of $\mathrm{rank}(\mathcal{E}) = r$ on X.
(1) $\mathcal{E}$ is said to be *ample and spanned* if the tautological line bundle $H(\mathcal{E})$ of $\mathbb{P}_X(\mathcal{E})$ is ample and spanned.
(2) $\mathcal{E}$ is said to be *very ample* if the tautological line bundle $H(\mathcal{E})$ of $\mathbb{P}_X(\mathcal{E})$ is very ample.

Remark 2.1 Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle of $\mathrm{rank}(\mathcal{E}) = r$ on X.
(1) Assume that $n-r \geq 1$ and $\mathcal{E}$ is spanned. Then there exists an element $s \in H^0(\mathcal{E})$ such that the zero locus of s is a submanifold of X of dimension $n-r$.
(2) If $\mathcal{E}$ is very ample, then $\mathcal{E}$ is ample and spanned.
(3) Let $\mathcal{E}$ be a very ample (resp. ample and spanned) vector bundle on X and let $\mathcal{F}$ be a quotient bundle of $\mathcal{E}$. Then $\mathcal{F}$ is also very ample (resp. ample and spanned).

Theorem 2.2 *Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Assume that $1 \leq r \leq n-1$ and $\mathcal{E}$ is spanned. Let Z be a zero locus of a general section of $H^0(\mathcal{E})$. Then $g_i(X, \mathcal{E}) = g_i(Z, c_1(\mathcal{E}|_Z))$ for every integer i with $0 \leq i \leq n-r$.*

Proof. See [8], Theorem 2.2. □

Theorem 2.3 (Lefschetz-Sommese) *Let X be an n-dimensional smooth projective variety, and let $\mathcal{E}$ be an ample vector bundle of $\mathrm{rank}(\mathcal{E}) = r \geq 2$ on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is a submanifold of X of the expected dimension $n-r$. Let $r_q : H^q(X, \mathbb{Z}) \to H^q(Z, \mathbb{Z})$ be the restriction homomorphism. Then*

(1) *r_q is an isomorphism for $q \leq n-r-1$.*

(2) *r_q is injective and its cokernel is torsion free for $q = n-r$.*

Proof. See [12], 1.3 Theorem. □

Remark 2.2 Let X, $\mathcal{E}$, and Z be as in Theorem 2.3. By the Hodge theory, we obtain that $h^q(\mathcal{O}_X) = h^q(\mathcal{O}_Z)$ for every integer q with $0 \leq q \leq n-r-1$, and $h^{n-r}(\mathcal{O}_X) \leq h^{n-r}(\mathcal{O}_Z)$.

Theorem 2.4 *Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Assume that $1 \leq r \leq n$ and $\mathcal{E}$ is spanned. Then $g_i(X, \mathcal{E}) \geq h^i(\mathcal{O}_X)$ for $0 \leq i \leq n-r$.*

Proof. See [8], Corollary 2.6. □

Theorem 2.5 *Let X be a smooth projective variety of $\dim X = n \geq 3$ and let $\mathcal{E}$ be a very ample vector bundle of rank $r \geq 2$ on X. Then $g_2(X, \det(\mathcal{E})) = h^2(\mathcal{O}_X) + 1$ if and only if $(X, \mathcal{E})$ is one of the following:*

(1) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2})$.

(2) $(\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})$.

(3) *$X \cong \mathbb{P}^3$, and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$, $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2)$, $\mathcal{T}_{\mathbb{P}^3}$, $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$, $\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}$, or $\mathcal{N}(2)$, where $\mathcal{N}$ is the null-correlation bundle on $\mathbb{P}^3$.*

(4) *$X \cong \mathbb{Q}^3$, and $\mathcal{E} \cong \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3}, \mathcal{O}_{\mathbb{Q}^3}(1) \oplus \mathcal{O}_{\mathbb{Q}^3}(2)$, or $\mathcal{S}(2)$, where $\mathcal{S}$ is the Spinor bundle on $\mathbb{Q}^3$.*

(5) *$X \cong \mathbb{P}^2 \times \mathbb{P}^1$, and $\mathcal{E} \cong \mathcal{O}(2,1) \oplus \mathcal{O}(1,1)$ or $p_1^*\mathcal{T}_{\mathbb{P}^2} \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, where p_i is the i-th projection for $i = 1,2$ and $\mathcal{O}(a,b) := p_1^*\mathcal{O}_{\mathbb{P}^2}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(b)$.*

(6) *$(X, A^{\oplus 2})$, where (X, A) is a Del Pezzo 3-fold of degree d ($3 \le d \le 7$).*

(7) *$n = 3$ and there exists a fibration $f : X \to W$ over a smooth elliptic curve W such that $(F, \mathcal{E}_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ for every fiber F of f.*

(8) *$n = 3$ and there exists a fibration $f : X \to W$ over a smooth elliptic curve W such that $(F, \mathcal{E}_F) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$ for a general fiber F of f.*

Proof. See [11]. □

3 Main Theorem.

Theorem 3.1 *Let $(X, \mathcal{E})$ be a generalized polarized manifold of $\dim X = n \ge 3$ and $\operatorname{rank}(\mathcal{E}) = r \ge 2$. Assume that $n - r \ge 3$ and $\mathcal{E}$ is very ample. If $g_2(X, \mathcal{E}) = h^2(\mathcal{O}_X) + 1$, then $(X, \mathcal{E})$ is one of the following:*

(a) $(\mathbb{P}^7, \mathcal{O}_{\mathbb{P}^7}(1)^{\oplus 2})$.

(b) $(\mathbb{P}^7, \mathcal{O}_{\mathbb{P}^7}(1)^{\oplus 4})$.

(c) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2) \oplus \mathcal{O}_{\mathbb{P}^5}(1))$.

(d) $(\mathbb{Q}^6, \mathcal{O}_{\mathbb{Q}^6}(1)^{\oplus 3})$.

(e) $(\mathbb{Q}^6, \mathcal{O}_{\mathbb{Q}^6}(1)^{\oplus 2})$.

(f) *X is a 5-dimensional Fano manifold of index 4 and $r = 2$. Moreover $\operatorname{Pic}(X) \cong \mathbb{Z} \cdot H$ and $\mathcal{E}_l \cong H_l^{\oplus 2}$ for every line l of (X, H).*

(g) *There exists a surjective morphism $f : X \to W$ over a smooth elliptic curve W such that a general fiber of f is a smooth hyperquadric $\mathbb{Q}^4$ with $\mathcal{E}|_F \cong \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2}$.*

(h) *There exists a surjective morphism $f : X \to W$ over a smooth elliptic curve W such that a general fiber F of f is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{G})$ for some vector bundle $\mathcal{G}$ of rank 4 on $\mathbb{P}^1$ and $\mathcal{E}|_F = \oplus_{j=1}^2 (H(\mathcal{G}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, where $H(\mathcal{G})$ is the tautological line bundle of $\mathcal{G}$ and $\pi : F \to \mathbb{P}^1$ is the bundle projection.*

Proof. By assumption, there exists a section $s \in H^0(\mathcal{E})$ such that the zero locus $Z := (s)_0$ is a smooth projective variety of $\dim Z = n - r \geq 3$. Then by Theorem 2.2 and Theorem 2.3 we get that

$$\begin{aligned} g_2(Z, c_1(\mathcal{E}|_Z)) &= g_2(X, \mathcal{E}) \\ &= h^2(\mathcal{O}_X) + 1 \\ &= h^2(\mathcal{O}_Z) + 1. \end{aligned}$$

Here we note that $\dim Z \geq 3$ by assumption. Hence by Theorem 2.5, $(Z, \mathcal{E}|_Z)$ is one of the following:

(I) $(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 2})$.

(II) $(\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})$.

(III) $Z \cong \mathbb{P}^3$, and $\mathcal{E}|_Z \cong \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$, $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2)$, $\mathcal{T}_{\mathbb{P}^3}$, $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$, $\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2}$, or $\mathcal{N}(2)$, where $\mathcal{N}$ is the null-correlation bundle on $\mathbb{P}^3$.

(IV) $Z \cong \mathbb{Q}^3$, and $\mathcal{E}|_Z \cong \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3}, \mathcal{O}_{\mathbb{Q}^3}(1) \oplus \mathcal{O}_{\mathbb{Q}^3}(2)$, or $\mathcal{S}(2)$, where $\mathcal{S}$ is the Spinor bundle on $\mathbb{Q}^3$.

(V) $Z \cong \mathbb{P}^2 \times \mathbb{P}^1$, and $\mathcal{E}|_Z \cong \mathcal{O}(2,1) \oplus \mathcal{O}(1,1)$ or $p_1^*\mathcal{T}_{\mathbb{P}^2} \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, where p_i is the i-th projection for $i = 1, 2$ and $\mathcal{O}(a,b) := p_1^*\mathcal{O}_{\mathbb{P}^2}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(b)$.

(VI) $(Z, A^{\oplus 2})$, where (Z, A) is a Del Pezzo 3-fold of degree d ($3 \leq d \leq 7$).

(VII) $n - r = 3$ and there exists a fibration $h : Z \to W$ over a smooth elliptic curve W such that $(F_h, \mathcal{E}_{F_h}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ for every fiber F_h of h.

(VIII) $n - r = 3$ and there exists a fibration $h : Z \to W$ over a smooth elliptic curve W such that $(F_h, \mathcal{E}_{F_h}) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$ for a general fiber F_h of h.

(A) Assume that $Z \cong \mathbb{P}^{n-r}$. Then $(Z, \mathcal{E}|_Z)$ is either (I) or (III).
Then by [12], Theorem A, we get that $(X, \mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$ since $n - r \geq 3$. We note that $h^2(\mathcal{O}_X) = 0$. We also note the following:

$$\begin{aligned} K_X &= \mathcal{O}_{\mathbb{P}^n}(-(n+1)), \\ c_2(X) &= \binom{n+1}{2} \mathcal{O}_{\mathbb{P}^n}(1)^2, \\ c_1(\mathcal{E}) &= r\mathcal{O}_{\mathbb{P}^n}(1), \\ c_2(\mathcal{E}) &= \binom{r}{2} \mathcal{O}_{\mathbb{P}^n}(1)^2, \\ c_r(\mathcal{E}) &= \mathcal{O}_{\mathbb{P}^n}(1)^r. \end{aligned}$$

If $(Z, \mathcal{E}|_Z)$ is the case (I) (resp. (III)), then $n - r = 5$ (resp. 3). Here we calculate $g_2(X, \mathcal{E})$ in this case.

Assume that $(Z, \mathcal{E}|_Z)$ is the case (III). In this case $n-r=3$ and $n \geq 5$. Then by Theorem 2.1 (2)

$$g_2(X,\mathcal{E}) = -1 + \frac{1}{12}(n-3)(2n^2-24n+76).$$

Since $g_2(X,\mathcal{E}) = h^2(\mathcal{O}_X)+1 = 1$, we obtain that $n=7$ and $r=4$. Namely $(X,\mathcal{E}) \cong (\mathbb{P}^7, \mathcal{O}(1)^{\oplus 4})$. This is the type (b) in Theorem 3.1.

Assume that $(Z, \mathcal{E}|_Z)$ is the case (I). In this case $n-r=5$ and $n \geq 7$. Then by Theorem 2.1 (2)

$$g_2(X,\mathcal{E}) = -1 + \frac{1}{4}(n-5)^3(5n^2-68n+232).$$

Since $g_2(X,\mathcal{E}) = h^2(\mathcal{O}_X)+1 = 1$, we obtain that $n=7$ and $r=2$. Namely $(X,\mathcal{E}) \cong (\mathbb{P}^7, \mathcal{O}_{\mathbb{P}^7}(1)^{\oplus 2})$. This is the type (a) in Theorem 3.1.
(B) Assume that $Z \cong \mathbb{Q}^{n-r}$. Then $(Z, \mathcal{E}|_Z)$ is either (II) or (IV).
Then by [12], Theorem B, we get that $(X,\mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r-1})$ or $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus r})$ since $n-r \geq 3$. We note that $h^2(\mathcal{O}_X)=0$.

If $(Z, \mathcal{E}|_Z)$ is the case (IV) (resp. (II)), then $n-r=3$ (resp. 4). Here we calculate $g_2(X,\mathcal{E})$ in this case.

Assume that $(X,\mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r-1})$. We also note the following:

$$\begin{aligned}
K_X &= \mathcal{O}_{\mathbb{P}^n}(-(n+1)), \\
c_2(X) &= \binom{n+1}{2}\mathcal{O}_{\mathbb{P}^n}(1)^2, \\
c_1(\mathcal{E}) &= (r+1)\mathcal{O}_{\mathbb{P}^n}(1), \\
c_2(\mathcal{E}) &= \left(2(r-1) + \binom{r-1}{2}\right)\mathcal{O}_{\mathbb{P}^n}(1)^2, \\
c_r(\mathcal{E}) &= 2\mathcal{O}_{\mathbb{P}^n}(1)^r.
\end{aligned}$$

If $(Z, \mathcal{E}|_Z)$ is the case (IV), then $n-r=3$ and $n \geq 5$. By Theorem 2.1 (2)

$$g_2(X,\mathcal{E}) = -1 + \frac{1}{6}(n-2)(2n^2-17n+39).$$

Since $g_2(X,\mathcal{E}) = h^2(\mathcal{O}_X)+1 = 1$, we obtain that $n=5$ and $r=2$. Namely $(X,\mathcal{E}) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2) \oplus \mathcal{O}_{\mathbb{P}^5}(1))$. This is the type (c) in Theorem 3.1.

If $(Z, \mathcal{E}|_Z)$ is the case (II), then $n-r=4$ and $n \geq 6$. By Theorem 2.1 (2)

$$g_2(X,\mathcal{E}) = -1 + \frac{1}{6}(n-3)^2\left(7n^2-66n+158\right).$$

But in this case $g_2(X,\mathcal{E}) \neq h^2(\mathcal{O}_X)+1$.

Assume that $(X,\mathcal{E})=(\mathbb{Q}^n,\mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus r})$. We also note the following:

$$\begin{aligned} K_X &= \mathcal{O}_{\mathbb{Q}^n}(-n),\\ c_2(X) &= \left(\binom{n+2}{2}-2n\right)\mathcal{O}_{\mathbb{Q}^n}(1)^2,\\ c_1(\mathcal{E}) &= r\mathcal{O}_{\mathbb{Q}^n}(1),\\ c_2(\mathcal{E}) &= \binom{r}{2}\mathcal{O}_{\mathbb{Q}^n}(1)^2,\\ c_r(\mathcal{E}) &= \mathcal{O}_{\mathbb{Q}^n}(1)^r. \end{aligned}$$

If $(Z,\mathcal{E}|_Z)$ is the case (IV), then $n-r=3$ and $n\geq 5$. By Theorem 2.1 (2)

$$g_2(X,\mathcal{E})=-1+\frac{1}{6}(n-3)(2n^2-21n+58).$$

Since $g_2(X,\mathcal{E})=h^2(\mathcal{O}_X)+1=1$, we obtain that $n=6$ and $r=3$. Namely $(X,\mathcal{E})\cong(\mathbb{Q}^6,\mathcal{O}_{\mathbb{Q}^6}(1)^{\oplus 3})$. This is the type (d) in Theorem 3.1.

If $(Z,\mathcal{E}|_Z)$ is the case (II), then $n-r=4$ and $n\geq 6$. By Theorem 2.1 (2)

$$g_2(X,\mathcal{E})=-1+\frac{1}{6}(n-4)^2(7n^2-80n+231).$$

Since $g_2(X,\mathcal{E})=h^2(\mathcal{O}_X)+1=1$, we obtain that $n=6$ and $r=2$. Namely $(X,\mathcal{E})\cong(\mathbb{Q}^6,\mathcal{O}_{\mathbb{Q}^6}(1)^{\oplus 2})$. This is the type (e) in Theorem 3.1.

(C) Assume that $Z\cong\mathbb{P}^2\times\mathbb{P}^1$. Then by Theorem 2.3, $H^i(X,\mathbb{Z})\cong H^i(Z,\mathbb{Z})$ for $i=1,2$. By the Hodge theory, we obtain that $h^i(\mathcal{O}_X)=h^i(\mathcal{O}_Z)$ for $i=1,2$. Hence $\rho:\mathrm{Pic}(X)\to\mathrm{Pic}(Z)$ is an isomorphism by the following commutative diagram:

$$\begin{array}{ccccccccc} H^1(X,\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_X)\\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow & & \downarrow\\ H^1(Z,\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_Z) & \longrightarrow & \mathrm{Pic}(Z) & \longrightarrow & H^2(Z,\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_Z) \end{array}$$

We take $p_1^*\mathcal{O}_{\mathbb{P}^2}(1)\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)\in\mathrm{Pic}(Z)$, where p_i is the i-th projection for $i=1,2$. Then there exists $H\in\mathrm{Pic}(X)$ such that $H|_Z=p_1^*\mathcal{O}_{\mathbb{P}^2}(1)\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then by considering the second projection $p_2:Z\to\mathbb{P}^1$, we obtain that $(Z,H|_Z)$ is a scroll over $\mathbb{P}^1$. Hence by [13] Theorem B, (X,H) is a scroll over $\mathbb{P}^1$ such that $\mathcal{E}|_F\cong\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus r}$ for every fiber F of $f:X\to\mathbb{P}^1$ and $f|_Z=p_2$. In particular $\mathcal{E}|_{F_Z}$ is split, where F_Z is a fiber of $p_2:Z\to\mathbb{P}^1$. On the other hand since $\mathcal{E}|_Z\cong\mathcal{O}(2,1)\oplus\mathcal{O}(1,1)$ or $p_1^*\mathcal{T}_{\mathbb{P}^2}\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, we obtain that $r=2$, and $\mathcal{E}|_{F_Z}\cong\mathcal{O}_{\mathbb{P}^2}(2)\oplus\mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{T}_{\mathbb{P}^2}$. Since $\mathcal{E}|_{F_Z}$ is split, we get that $\mathcal{E}|_{F_Z}\cong\mathcal{O}_{\mathbb{P}^2}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$. But since $\mathcal{E}|_F\cong\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus 2}$, this is a contradiction. Hence this case cannot occur.

(D) Assume that $(Z, \mathcal{E}|_Z) \cong (Z, A^{\oplus 2})$, where (Z, A) is a Del Pezzo 3-fold of degree d, where d is an integer with $3 \le d \le 7$.

In this case $n - r = 3$ and $r = 2$. Namely $n = 5$ and $r = 2$.

(D.1) If $\rho(Z) = 1$, then by [12], 2.5 Proposition, we get the following:

(D.1.1) $X = \mathbb{P}^5$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^5}(3) \oplus \mathcal{O}_{\mathbb{P}^5}(1)$ or $\mathcal{E}$ has the generic splitting type $(2, 2)$.

(D.1.2) $X = \mathbb{Q}^5$ and $\mathcal{E}_l = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ for every line $l \subset \mathbb{Q}^5$.

(D.1.3) X is a 5-dimensional Fano manifold of index 4 and $r = 2$. Moreover $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H$ and $\mathcal{E}_l \cong H_l^{\oplus 2}$ for every line l of (X, H).

Claim 3.1 *The cases* (D.1.1) *and* (D.1.2) *are impossible.*

Proof. By [12], 2.4, we get that $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$, A is the ample generator of $\mathrm{Pic}(Z)$, and $H_Z = A$, where H is the ample generator of $\mathrm{Pic}(X)$.

First we assume that $(X, \mathcal{E})$ is the case (D.1.1). Then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^5}(4)$. On the other hand $c_1(\mathcal{E}) = 2\mathcal{O}_{\mathbb{P}^5}(1)$ because $c_1(\mathcal{E}|_Z) = 2A$. But this is a contradiction.

Next we assume that $(X, \mathcal{E})$ is the case (D.1.2). Then since $c_1(\mathcal{E}|_Z) = 2A$, we obtain that $c_1(\mathcal{E}) = 2\mathcal{O}_{\mathbb{Q}^5}(1)$. We put $\mathcal{O}_{\mathbb{P}^1}(a) := \mathcal{O}_{\mathbb{Q}^5}(1)|_l$ for a line $l \subset \mathbb{Q}^5$. Then $c_1(\mathcal{E})|_l = \mathcal{O}_{\mathbb{P}^1}(2a)$. On the other hand $c_1(\mathcal{E})|_l = \mathcal{O}_{\mathbb{P}^1}(3)$ by assumption of (D.1.2). Hence $2a = 3$. But this is impossible because $a \in \mathbb{Z}$. This completes the proof of Claim 3.1. □

If $(X, \mathcal{E})$ is the case (D.1.3), then we get the type (f) in Theorem 3.1.

(D.2) If $\rho(Z) \ge 2$, then by [3], Theorem 1, we obtain the following:
There exist a smooth projective surface S and an ample vector bundle $\mathcal{F}$ of rank 4 on S such that $X = \mathbb{P}_S(\mathcal{F})$, where

$$S \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \text{if } Z \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \\ \mathbb{P}^2 & \text{otherwise.} \end{cases}$$

Moreover $\mathcal{E} = H(\mathcal{F}) \otimes f^*(\mathcal{G})$, where $H(\mathcal{F})$ is the tautological line bundle of $\mathcal{F}$ on X, $f : X \to S$ is the bundle projection and $\mathcal{G}$, a vector bundle of rank 2 on S, is the dual of the kernel of the vector bundle surjection $\mathcal{F} \to \mathcal{B}$ corresponding to the fiberwise inclusion of $Z = \mathbb{P}_S(\mathcal{B})$ into X.

Claim 3.2 $\mathcal{E} = (H(\mathcal{F}) \otimes f^*(B))^{\oplus 2}$ *for some line bundle* $B \in \mathrm{Pic}(S)$.

Proof. First we note that $f|_Z = p$ and $H(\mathcal{F})|_Z = H(\mathcal{B})$. Since $\mathcal{E} = H(\mathcal{F}) \otimes f^*(\mathcal{G})$, we get that $\mathcal{E}|_Z = H(\mathcal{F})|_Z \otimes (f^*(\mathcal{G}))_Z \cong H(\mathcal{B}) \otimes p^*(\mathcal{G})$, where $p : Z = \mathbb{P}_S(\mathcal{B}) \to S$ is the projection. Hence $c_1(\mathcal{E}|_Z) = 2H(\mathcal{B}) + c_1(p^*(\mathcal{G}))$. Therefore $A_Z = H(\mathcal{B}) \otimes p^*(B)$ for some $B \in \mathrm{Pic}(S)$. Since $\mathcal{E}_Z = A_Z \oplus A_Z$, we obtain that $p^*(\mathcal{G}) = (H(\mathcal{B})^{-1} \otimes A_Z)^{\oplus 2} = (p^*(B))^{\oplus 2}$. Therefore $\mathcal{G} \cong B \oplus B$ and $\mathcal{E} = H(\mathcal{F}) \otimes f^*(\mathcal{G}) = (H(\mathcal{F}) \otimes f^*(B))^{\oplus 2}$. This completes the proof of Claim 3.2. □

Next we give the formula of $g_2(X,\mathcal{E})$. We note the following:

$$\begin{aligned} K_X &= -4H(\mathcal{F})+f^*(K_S+c_1(\mathcal{F})),\\ c_2(X) &= c_2(f^*\mathcal{T}_S)+c_1(f^*\mathcal{F}^\vee\otimes H(\mathcal{F}))c_1(f^*\mathcal{T}_S)+c_2(f^*\mathcal{F}^\vee\otimes H(\mathcal{F})),\\ c_1(\mathcal{E}) &= 2H(\mathcal{F})+2f^*(B),\\ c_2(\mathcal{E}) &= (H(\mathcal{F})+f^*(B))^2. \end{aligned}$$

By Theorem 2.1 (2), we get that following:

$$\begin{aligned} (1)\qquad & g_2(X,\mathcal{E})\\ &= -1+h^1(\mathcal{O}_X)+\frac{1}{12}\left\{2(K_S+c_1(\mathcal{F}))^2+48(K_S+c_1(\mathcal{F}))B\right.\\ &\quad +96B^2+8(K_S+c_1(\mathcal{F}))c_1(\mathcal{F})-4c_1(\mathcal{F})^2+2c_2(S)\\ &\quad \left.+6c_1(S)c_1(\mathcal{F})+24c_1(S)B\right\}. \end{aligned}$$

Here we note that $h^1(\mathcal{O}_X)=0$ in this case.

Claim 3.3 *Assume that $S\cong\mathbb{P}^2$. Then this case cannot occur.*

Proof. We put $c_1(\mathcal{F})=\mathcal{O}_{\mathbb{P}^2}(f)$ and $B=\mathcal{O}_{\mathbb{P}^2}(b)$, where f and b are integers. Then by (1) above,

$$g_2(X,\mathcal{E})=-1+\frac{1}{2}(f^2+8fb+16b^2-3f-12b+4).$$

If $g_2(X,\mathcal{E})=h^2(\mathcal{O}_X)+1$, then $f^2+8fb+16b^2-3f-12b=0$. Namely $(f+4b)(f+4b-3)=0$. Hence $f+4b=0$ or $f+4b=3$.

Since $\mathcal{E}$ is ample, $H(\mathcal{F})\otimes f^*(B)$ is also ample by Claim 3.2. We put $\mathcal{H}:=f_*(H(\mathcal{F})\otimes f^*(B))$. Then $X\cong\mathbb{P}_{\mathbb{P}^2}(\mathcal{H})$ and $H(\mathcal{H})=H(\mathcal{F})\otimes f^*(B)$.

Since $H(\mathcal{F})\otimes f^*(B)$ is ample, so is $H(\mathcal{H})$. Hence $\mathcal{H}$ is ample. Here we note that $\mathcal{H}=f_*(H(\mathcal{F})\otimes f^*(B))=\mathcal{F}\otimes B$. Then $c_1(\mathcal{H})=c_1(\mathcal{F})+4B=\mathcal{O}_{\mathbb{P}^2}(f+4b)$. Since $\mathcal{H}$ is ample, $f+4b>0$ and we obtain that $f+4b=3$.

Let l be a line in $\mathbb{P}^2$. Then $c_1(\mathcal{H})l=f+4b=3$. But since $\operatorname{rank}(\mathcal{H})=4$ and $l\cong\mathbb{P}^1$, we obtain that $c_1(\mathcal{H})l\geq 4$, and this is a contradiction. This completes the proof of Claim 3.3. □

Next we consider the case where $S\cong\mathbb{P}^1\times\mathbb{P}^1$.

Claim 3.4 *Assume that $S\cong\mathbb{P}^1\times\mathbb{P}^1$. Then this case cannot occur.*

Proof. First we note that for any member $D\in\operatorname{Pic}(\mathbb{P}^1\times\mathbb{P}^1)$, we can write $D=p_1^*(\mathcal{O}(a))\otimes p_2^*(\mathcal{O}(b))$ for some integers a and b, where $p_i:\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ is the i-th projection. We put $\mathcal{O}(a,b):=p_1^*(\mathcal{O}(a))\otimes p_2^*(\mathcal{O}(b))$. We also note that $c_1(\mathcal{B})=\mathcal{O}(2t_1,2t_2)$ for some integers t_1 and t_2 because $K_Z=-2H(\mathcal{B})+(f|_Z)^*(K_{\mathbb{P}^1\times\mathbb{P}^1}+c_1(\mathcal{B}))$, $K_{\mathbb{P}^1\times\mathbb{P}^1}=\mathcal{O}(-2,-2)$, and $K_Z=2D$ for some $D\in\operatorname{Pic}(Z)$. Since $(Z,H(\mathcal{B})\otimes(f|_Z)^*(B))$ is a Del Pezzo manifold, we obtain that $2(H(\mathcal{B})+(f|_Z)^*(B))=-K_Z=$

$2H(\mathcal{B})+(f|_Z)^*(\mathcal{O}(2-2t_1, 2-2t_2)$. We put $B=\mathcal{O}(b_1,b_2)$. Then we get that $t_i+b_i=1$ for $i=1,2$. On the other hand, by the following exact sequence

$$0 \to \mathcal{G}^\vee \to \mathcal{F} \to \mathcal{B} \to 0,$$

we obtain that $c_1(\mathcal{F}) = c_1(\mathcal{G}^\vee)+c_1(\mathcal{B})$. Hence $c_1(\mathcal{F})=\mathcal{O}(2t_1-2b_1, 2t_2-2b_2)$ because $\mathcal{G}\cong B\oplus B$. Since $t_i+b_i=1$ for $i=1,2$, we obtain that

$$c_1(\mathcal{F}) = \mathcal{O}(2-4b_1, 2-4b_2). \tag{2}$$

Since $\mathcal{F}$ is ample, $b_1 \le 0$ and $b_2 \le 0$ are obtained.

Next we calculate $g_2(X,\mathcal{E})$ by using b_1 and b_2. We note that

$$K_S = \mathcal{O}(-2,-2) \tag{3}$$

$$c_2(S) = 4. \tag{4}$$

By (1), (2), (3), and (4), we obtain that $g_2(X,\mathcal{E}) = 1-4b_1-4b_2$. Since $g_2(X,\mathcal{E}) = h^2(\mathcal{O}_X)+1=1$, we get that $b_1=b_2=0$ and $c_1(\mathcal{F})=\mathcal{O}(2,2)$. Let F' be a fiber of the first projection $\mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1$. Then $c_1(\mathcal{F})F'=2$. But since $\mathcal{F}$ is ample, $\operatorname{rank}(\mathcal{F})=4$, and $F'\cong\mathbb{P}^1$, we obtain that $c_1(\mathcal{F})F' \ge 4$. This is a contradiction. This completes the proof of Claim 3.4. □

(E) Assume that $n-r=3$ and there exists a fibration $h: Z\to W$ over a smooth elliptic curve W such that $(F_h, \mathcal{E}|_{F_h}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(2))$ for every fiber F_h of h or $(F_h,\mathcal{E}|_{F_h})\cong(\mathbb{Q}^2,\mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$ for a general fiber F_h of h.

In this case, we have $n=5$ and $r=2$.

Claim 3.5 *Let $\alpha: X\to\alpha(X)$ be the Albanese map of X. Then $\alpha(X)$ is a smooth elliptic curve, $W\cong\alpha(X)$, and $h=\alpha|_Z$.*

Proof. Since $h^1(\mathcal{O}_X)=h^1(\mathcal{O}_Z)=1$, we obtain that $\alpha(X)=\operatorname{Alb}(X)$ and $\alpha(X)$ is a smooth elliptic curve. (Here $\operatorname{Alb}(X)$ denotes the Albanese variety of X.) Let $\alpha|_Z: Z\to\alpha(X)$. Then $\alpha|_Z$ is surjective. Here we note that $h: Z\to W$ is a surjective morphism with connected fibers such that a general fiber F_h is $\mathbb{P}^2$ or $\mathbb{Q}^2$. Hence $\alpha|_Z(F_h)$ is a point. Therefore by [2], Lemma 4.1.13, there exists a surjective morphism $\delta: W\to\alpha(X)$ such that $\alpha|_Z=\delta\circ h$. But since h has connected fibers, δ is an isomorphism. □

Let F_h (resp. F_α) be a general fiber of h (resp. α). Since $K_{F_h}+c_1(\mathcal{E}|_{F_h})=\mathcal{O}_{F_h}$, $h=\alpha|_Z$, and $Z\cap F_\alpha=F_h$, we obtain that

$$((K_X+2c_1(\mathcal{E}))|_{F_\alpha})|_{F_h} \cong ((K_X+2c_1(\mathcal{E}))|_Z)|_{F_\alpha} \cong (K_Z+c_1(\mathcal{E}|_Z))|_{F_h} \cong \mathcal{O}_{F_h}. \tag{$\star$}$$

Here we note that since Z is the zero locus of a general member of $H^0(\mathcal{E})$, a general fiber F_h of $h: Z\to W$ is the zero locus of a general member of $H^0(\mathcal{E}|_{F_\alpha})$ by Claim 3.5.

(E.1) If $F_h=\mathbb{P}^2$, then by [12], Theorem A, we get that $(F_\alpha,\mathcal{E}|_{F_\alpha})\cong(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2})$. In particular, $(K_X+2c_1(\mathcal{E}))|_{F_\alpha}\cong\mathcal{O}_{\mathbb{P}^4}(-1)$. But by [$\star$], this is impossible.

(E.2) Assume that $(F_h,\mathcal{E}|_{F_h})\cong(\mathbb{Q}^2,\mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$. By [12], Theorem B, $(F_\alpha,\mathcal{E}|_{F_\alpha})$ is one of the following:

(E.2.1) $(F_\alpha, \mathcal{E}|_{F_\alpha}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$.

(E.2.2) $(F_\alpha, \mathcal{E}|_{F_\alpha}) \cong (\mathbb{Q}^4, \mathcal{O}_{\mathbb{Q}^4}(1)^{\oplus 2})$.

(E.2.3) $F_\alpha = \mathbb{P}_{\mathbb{P}^1}(\mathcal{G})$ for some vector bundle $\mathcal{G}$ of rank 4 on $\mathbb{P}^1$ and $\mathcal{E}|_{F_\alpha} = \oplus_{j=1}^2 (H(\mathcal{G}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, where $H(\mathcal{G})$ is the tautological line bundle of $\mathcal{G}$ and $\pi : F_\alpha \to \mathbb{P}^1$ is the bundle projection.

If $(F_\alpha, \mathcal{E}|_{F_\alpha})$ is the case (E.2.1), then $(K_X + 2c_1(\mathcal{E}))|_{F_\alpha} \cong \mathcal{O}_{\mathbb{P}^4}(1)$, and this is impossible by $[\star]$.

By putting $f := \alpha$, the case (E.2.2) (resp. (E.2.3)) is the type (g) (resp. (h)) in Theorem 3.1.

These complete the proof of Theorem 3.1. □

Example 3.1 Here we will give an example of the case (f) in Theorem 3.1.

Let (X, H) be a 5-dimensional Fano manifold of index 4 with $H^5 \geq 3$ and let $\mathcal{E} = H^{\oplus 2}$. Then $\mathcal{E}$ is a very ample vector bundle of rank 2 and

$$\begin{aligned} c_1(\mathcal{E}) &= 2H, \\ c_2(\mathcal{E}) &= H^2. \end{aligned}$$

On the other hand

$$\begin{aligned} K_X &= -4H, \\ c_2(X)H^3 &= 12 + 5H^5. \end{aligned}$$

Hence by the definition of the second c_r-sectional geometric genus, we obtain that $g_2(X, \mathcal{E}) = 1 = h^2(\mathcal{O}_X) + 1$.

Problem 3.1 *Does there exist a very ample vector bundle $\mathcal{E}$ of* $\operatorname{rank}(\mathcal{E}) = 2$ *on a Fano* 5*-fold X of index* 4 *such that $\mathcal{E}$ is not split and* $g_2(X, \mathcal{E}) = h^2(\mathcal{O}_X) + 1$ *?*

Example 3.2 Here we consider the case (g) in Theorem 3.1.

Let (X, L) be a hyperquadric fibration over a smooth elliptic curve C. Let $f : X \to C$ be its morphism. We put $\mathcal{F} := f_*(L)$. Then $\mathcal{F}$ is a locally free sheaf of $\operatorname{rank}(\mathcal{F}) = n + 1$, where $n = \dim X$. In this case there exists an embedding $\iota : X \to \mathbb{P}_C(\mathcal{F})$ such that $f = \pi \circ \iota$ and $X \in |2H(\mathcal{F}) + \pi^*(D)|$, where $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection, $H(\mathcal{F})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{F})$, and $D \in \operatorname{Pic}(C)$. Here we assume that $n = 5$ and we put $\mathcal{E} := L \oplus L$. Then $\mathcal{E}$ is an ample vector bundle of $\operatorname{rank}(\mathcal{E}) = 2$ on X.

Next we calculate $g_2(X, \mathcal{E})$. We note the following:

$$\begin{aligned} H(\mathcal{F})|_X &= L \\ c_1(\mathcal{E}) &= 2L \\ c_2(\mathcal{E}) &= L^2 \\ K_X &= -4L + f^*(c_1(\mathcal{F}) + D) \\ c_2(X) &= 7L^2 - 3Lf^*(c_1(\mathcal{F})) - 2Lf^*(D). \end{aligned}$$

We put $b = \deg D$ and $e = \deg \mathcal{F}$. By Theorem 2.1 (2) and the above equalities, we obtain that $g_2(X, \mathcal{E}) = b + e$. On the other hand the sectional genus of (X, L) $g_1(X, L) = 1 + b + e$ (see [4]). By [4], Example 3.9 and Example 3.11, there exists a hyperquadric fibration (X, L) over a smooth elliptic curve C with $\dim X = 5$ such that $(b, e, L^5) = (1, 0, 1)$ or $(0, 1, 2)$. In these cases $g_2(X, \mathcal{E}) = 1 = h^2(\mathcal{O}_X) + 1$.

But we note that $\mathcal{E}$ is not very ample in each case. First we can prove that L is not very ample. (If L is very ample, then $X \cong \mathbb{P}^5$ or $\mathbb{Q}^5$ because $L^5 = 1$ or 2. But this is impossible because $\mathrm{Pic}(X) \cong \mathbb{Z}$ in each case.) Therefore $\mathcal{E}$ is not very ample because $\mathcal{E} \to L$ is surjective. (See Remark 2.1 (3).)

The existence of the case (h) in Theorem 3.1 is uncertain at present.

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